

ON CONNECTEDNESS OF SETS IN THE REAL SPECTRA OF POLYNOMIAL RINGS

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Abstract

Let R be a real closed field. The Pierce-Birkhoff conjecture says that any piecewise polynomial function f on R^n can be obtained from the polynomial ring $R[x_1, \dots, x_n]$ by iterating the operations of maximum and minimum. The purpose of this paper is threefold. First, we state a new conjecture, called the Connectedness conjecture, which asserts, for every pair of points $\alpha, \beta \in \text{Sper } R[x_1, \dots, x_n]$, the existence of connected sets in the real spectrum of $R[x_1, \dots, x_n]$, satisfying certain conditions. We prove that the Connectedness conjecture implies the Pierce-Birkhoff conjecture.

Secondly, we construct a class of connected sets in the real spectrum which, though not in itself enough for the proof of the Pierce-Birkhoff conjecture, is the first and simplest example of the sort of connected sets we really need, and which constitutes the first step in our program for a proof of the Pierce-Birkhoff conjecture in dimension greater than 2.

Thirdly, we apply these ideas to give two proofs of the Connectedness conjecture (and hence also of the Pierce-Birkhoff conjecture in the abstract formulation) in the special case when one of the two points $\alpha, \beta \in \text{Sper } R[x_1, \dots, x_n]$ is monomial. One of the proofs is elementary while the other consists in deducing the (monomial) Connectedness conjecture as an immediate corollary of the main connectedness theorem of this paper.

1 Introduction

All the rings in this paper will be commutative with 1. Throughout this paper, R will denote a real closed field and A the polynomial ring $R[x_1, \dots, x_n]$, unless otherwise specified.

The Pierce-Birkhoff conjecture asserts that any piecewise-polynomial function $f : R^n \rightarrow R$ can be expressed as a maximum of minima of a finite family of polynomials (see below for the definitions and a precise statement of the conjecture). This paper is the first step in our program for a proof of the Pierce-Birkhoff conjecture in its full generality (the best results up to now are due to Louis Mahé [15], who proved the conjecture for $n = 2$, as well as some partial results for $n = 3$).

We start by stating the Pierce-Birkhoff conjecture in its original form as it was first stated by M. Henriksen and J. Isbell in the early nineteen sixties.

Definition 1.1 A function $f : R^n \rightarrow R$ is said to be **piecewise polynomial** if R^n can be covered by a finite collection of closed semi-algebraic sets P_i such that for each i there exists a polynomial $f_i \in A$ satisfying $f|_{P_i} = f_i|_{P_i}$. Given a piecewise polynomial function f , we say that f is **defined by r polynomials** if r is the number of distinct polynomials among the f_i above.

Clearly, any piecewise polynomial function is continuous. Piecewise polynomial functions form a ring, containing A , which is denoted by $PW(A)$.

On the other hand, one can consider the (lattice-ordered) ring of all the functions obtained from A by iterating the operations of \sup and \inf . Since applying the operations of \sup and \inf to polynomials produces functions which are piecewise polynomial, this ring is contained in $PW(A)$ (the latter ring is closed under \sup and \inf). It is natural to ask whether the two rings coincide. The precise statement of the conjecture is:

Conjecture 1 (Pierce–Birkhoff) *If $f : R^n \rightarrow R$ is in $PW(A)$, then there exists a finite family of polynomials $g_{ij} \in A$ such that $f = \sup_i \inf_j (g_{ij})$ (in other words, for all $x \in R^n$, $f(x) = \sup_i \inf_j (g_{ij}(x))$).*

A few words about the history of the problem. A question suggestive of the present Pierce–Birkhoff conjecture appeared in the 1956 paper of Birkhoff and Pierce [4], where f -rings were first defined. The question seems to have been phrased somewhat carelessly, and the intended meaning of the text is not easy to understand.

According to Mel Henriksen, he and Isbell attempted to prove the conjecture as it is now formulated while working on f -rings in the early nineteen sixties [8]. Since they obtained no significant results about it, they never mentioned it explicitly in print.

In the early nineteen eighties, when real algebraic geometry was becoming active, Isbell described the question to several mathematicians at several meetings. Efroymsen and Mahé took up the challenge and worked on it.

Isbell did not hesitate to read the 1956 question posed by Birkhoff and Pierce quite generously. He seems to be the one responsible for attaching the name “Pierce–Birkhoff conjecture” to the problem. He believed that Birkhoff and Pierce intended to ask the very question we now call the Pierce–Birkhoff conjecture.

It is not clear how or why the names Pierce and Birkhoff came to appear in reverse alphabetical order, rather than the alphabetical order that was used in their 1956 paper. In light of the history, it might be more accurate to call the question the “Pierce–Birkhoff–Isbell–Henriksen conjecture”.

In 1989 J.J. Madden [13] reformulated this conjecture in terms of the real spectrum of A and separating ideals. We now recall Madden’s formulation together with the relevant definitions.

Let B be a ring. A point α in the real spectrum of B is, by definition, the data of a prime ideal \mathfrak{p} of B , and a total ordering \leq of the quotient ring B/\mathfrak{p} , or, equivalently, of the field of fractions of B/\mathfrak{p} . Another way of defining the point α is as a homomorphism from B to a real closed field, where two homomorphisms are identified if they have the same kernel \mathfrak{p} and induce the same total ordering on B/\mathfrak{p} .

The ideal \mathfrak{p} is called the support of α and denoted by \mathfrak{p}_α , the quotient ring B/\mathfrak{p}_α by $B[\alpha]$, its field of fractions by $B(\alpha)$ and the real closure of $B(\alpha)$ by $k(\alpha)$. The total ordering of $B(\alpha)$ is denoted by \leq_α . Sometimes we write $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$.

Definition 1.2 *The real spectrum of B , denoted by $\text{Sper } B$, is the collection of all pairs $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$, where \mathfrak{p}_α is a prime ideal of B and \leq_α is a total ordering of B/\mathfrak{p}_α .*

The real spectrum $\text{Sper } B$ is endowed with two natural topologies. The first one, called the **spectral (or Harrison) topology**, has basic open sets of the form

$$U(f_1, \dots, f_k) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0\}$$

with $f_1, \dots, f_k \in B$. Here and below, we commit the following standard abuse of notation: for an element $f \in B$, $f(\alpha)$ stands for the natural image of f in $B[\alpha]$ and the inequality $f(\alpha) > 0$ really means $f(\alpha) >_\alpha 0$.

The second is the **constructible topology** whose basic open sets are of the form

$$V(f_1, \dots, f_k, g) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0, g(\alpha) = 0\},$$

where $f_1, \dots, f_k, g \in B$. Boolean combinations of sets of the form $V(f_1, \dots, f_n, g)$ are called **constructible sets** of $\text{Sper } B$.

Proposition 1.1 ([5], Proposition 7.1.12, p.114) *Let B be any ring, then $\text{Sper}(B)$ is compact for the constructible topology.*

Remark 1.1 *Since the spectral topology is coarser than the constructible topology, $\text{Sper}(B)$ is also compact for the spectral topology.*

Denote by $\text{Maxr}(A)$ the set of points $\alpha \in \text{Sper}(A)$ such that \mathfrak{p}_α is a maximal ideal of A . We view $\text{Maxr}(A)$ as a topological subspace of $\text{Sper}(A)$ with the spectral (respectively, constructible) topology. We may naturally identify R^n with $\text{Maxr}(A)$: a point $(a_1, \dots, a_n) \in R^n$ corresponds to the point $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha) \in \text{Sper}(A)$, where \mathfrak{p}_α is the maximal ideal

$$\mathfrak{p}_\alpha = (x_1 - a_1, \dots, x_n - a_n)$$

and \leq_α is the unique order on R . The spectral topology on $\text{Sper}(A)$ induces the euclidean topology on R^n , while the constructible topology induces the topology of R^n whose base consists of the semi-algebraic sets. The injection $R^n = \text{Maxr}(A) \hookrightarrow \text{Sper}(A)$ induces a one-to-one correspondence between the semi-algebraic sets of R^n and the constructible sets of $\text{Sper}(A)$: the set $S(f_1, \dots, f_k, g) = \{(a_1, \dots, a_n) \in R^n \mid f_1(a_1, \dots, a_n) > 0, \dots, f_k(a_1, \dots, a_n) > 0, g(a_1, \dots, a_n) = 0\}$ corresponds to the set

$$V(f_1, \dots, f_k, g) = \{\alpha \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0, g(\alpha) = 0\}.$$

For a general semi-algebraic set S in R^n , the corresponding constructible set of $\text{Sper}(A)$ will be denoted by \tilde{S} . Conversely, we sometimes start with a constructible set \tilde{S} in $\text{Sper}(A)$ and denote by S its semi-algebraic counterpart in R^n .

Notation: We will use the notation \coprod to denote the disjoint union of sets.

Recall Definition 2.4.2 and Proposition 7.5.1 of [5] (p. 31 and p. 130, respectively):

Definition 1.3 *Let S be a semi-algebraic subset of R^n . The set S is called semi-algebraically connected if, for any decomposition $S = S_1 \coprod S_2$, with S_1, S_2 semi-algebraic and relatively closed in S for the euclidean topology, we have either $S_1 = S$ or $S_2 = S$.*

Proposition 1.2 *A semi-algebraic set S in R^n is semi-algebraically connected if and only if \tilde{S} is connected in the spectral topology. If S_1, \dots, S_k are the semi-algebraically connected components of S , then $\tilde{S}_1, \dots, \tilde{S}_k$ are the connected components of \tilde{S} .*

Warning: Semi-algebraic connectedness is the same as the usual connectedness if $R = \mathbb{R}$, but not in general. For a general real closed field R , usual connectedness in the euclidean topology is a very restrictive notion and will not be used in this paper.

Proposition 1.3 ([5], Theorems 2.4.4 and 7.5.1). *Let $A = R[x_1, \dots, x_n]$ and let X be a constructible subset of $\text{Sper}(A)$. Then X can be written as*

$$X = \coprod_{i=1}^r X_i \quad (1)$$

where each X_i is constructible and connected in the spectral topology. In other words, X has finitely many connected components in the spectral topology, each of which is itself constructible.

Remark 1.2 *Let B be a finitely generated R -algebra. Then Proposition 1.3 holds also for $\text{Sper } A$ replaced with $\text{Sper } B$. Indeed, B is a homomorphic image of a polynomial ring of the form $R[x_1, \dots, x_n]$, so $\text{Sper } B$ is a closed subspace of $\text{Sper } R[x_1, \dots, x_n]$. Then a constructible subset X of $\text{Sper } B$ is also constructible viewed as a subset of $\text{Sper } R[x_1, \dots, x_n]$ and the decomposition (1) has the properties required in the Proposition regardless of whether one views the X_i as subsets of $\text{Sper } B$ or of $\text{Sper } R[x_1, \dots, x_n]$.*

Next, we recall the notion of **separating ideal**, introduced by Madden in [13].

Definition 1.4 *Let B be a ring. For $\alpha, \beta \in \text{Sper } B$, the **separating ideal** of α and β , denoted by $\langle \alpha, \beta \rangle$, is the ideal of B generated by all the elements $f \in B$ which change sign between α and β , that is, all the f such that $f(\alpha) \geq 0$ and $f(\beta) \leq 0$.*

We will need the following basic properties of the separating ideal, proved in [13]:

Proposition 1.4 *Let the notation be as above. We have:*

- (1) $\langle \alpha, \beta \rangle$ is both a ν_α -ideal and a ν_β -ideal.
- (2) α and β induce the same ordering on $\frac{B}{\langle \alpha, \beta \rangle}$ (in particular, the set of ν_α -ideals containing $\langle \alpha, \beta \rangle$ coincides with the set of ν_β -ideals containing $\langle \alpha, \beta \rangle$).
- (3) $\langle \alpha, \beta \rangle$ is the smallest ideal (in the sense of inclusion), satisfying (1) and (2).
- (4) If α and β have no common specialization then $\langle \alpha, \beta \rangle = B$.

Let f be a piecewise polynomial function on R^n and take a point $\alpha \in \text{Sper } A$. Let the notation be as in Definition 1.1. The covering $R^n = \bigcup_i P_i$ induces a corresponding covering $\text{Sper } A = \bigcup_i \tilde{P}_i$

of the real spectrum. Pick and fix an i such that $\alpha \in \tilde{P}_i$. We set $f_\alpha := f_i$. We refer to f_α as **a local polynomial representative of f at α** . In general, the choice of i is not uniquely determined by α . Implicit in the notation f_α is the fact that one such choice has been made.

In [13], Madden reduced the Pierce–Birkhoff conjecture to a purely local statement about separating ideals and the real spectrum. Namely, he showed that the Pierce–Birkhoff conjecture is equivalent to

Conjecture 2 (Pierce–Birkhoff conjecture, the abstract version) *Let f be a piecewise polynomial function and α, β points in $\text{Sper } A$. Let $f_\alpha \in A$ be a local representative of f at α and $f_\beta \in A$ a local representative of f at β . Then $f_\alpha - f_\beta \in \langle \alpha, \beta \rangle$.*

The following slightly weaker conjecture has proved to be an extremely useful stepping stone on the way to Pierce–Birkhoff (we give the contrapositive of Madden’s original statement, since it is better adapted to our needs).

Conjecture 3 (the Separation conjecture) *Let $g \in A$ and let $\alpha, \beta \in \text{Sper } A$ be two points such that $g \notin \langle \alpha, \beta \rangle$. Then α and β lie in the same connected component of the set $\text{Sper } A \setminus \{g = 0\}$.*

Proposition 1.5 *The Separation conjecture is equivalent to the special case of the Pierce–Birkhoff conjecture in which the piecewise-polynomial function f is defined by two polynomials.*

Proof: First, assume the Pierce–Birkhoff conjecture is true for any piecewise-polynomial f defined by two polynomials. Let g be as in the Separation conjecture. We argue by contradiction. Assume that α and β lie in two different connected components of $\text{Sper } A \setminus \{g = 0\}$. Let f be the piecewise polynomial function which is equal to g on the connected component of $\text{Sper } A \setminus \{g = 0\}$ containing α and $f = 0$ elsewhere. Then by the Pierce–Birkhoff conjecture we have $g - 0 = g \in \langle \alpha, \beta \rangle$, which gives the desired contradiction.

Conversely, assume that the Separation conjecture holds. Let f be a piecewise polynomial function defined by two distinct polynomials, f_1 and f_2 . Let P_1 and P_2 be closed semialgebraic sets such that $f|_{P_i} = f_i|_{P_i}$, $i \in \{1, 2\}$. Let $g = f_1 - f_2$. Then the interior of P_1 is disjoint from the interior of P_2 and each is a union of connected components of $\text{Sper } A \setminus \{g = 0\}$. Pick a point α in the interior of P_1 and β in the interior of P_2 . Then α and β lie in two different connected components of $\text{Sper } A \setminus \{g = 0\}$, hence $f_1 - f_2 \notin \langle \alpha, \beta \rangle$ by the separation conjecture. Thus the Pierce–Birkhoff conjecture holds for f . \square

We now state

Conjecture 4 (the Connectedness conjecture) *Let $\alpha, \beta \in \text{Sper } A$ and let g_1, \dots, g_s be a finite collection of elements of $A \setminus \langle \alpha, \beta \rangle$. Then there exists a connected set $C \subset \text{Sper } A$ such that $\alpha, \beta \in C$ and $C \cap \{g_i = 0\} = \emptyset$ for $i \in \{1, \dots, s\}$ (in other words, α and β belong to the same connected component of the set $\text{Sper } A \setminus \{g_1 \dots g_s = 0\}$).*

One advantage of the Connectedness conjecture is that it is a statement about polynomials which makes no mention of piecewise polynomial functions.

The purpose of this paper is threefold. First, we prove (§2) that the Connectedness conjecture implies the Pierce–Birkhoff conjecture. This reduces the Pierce–Birkhoff conjecture to constructing, for each $\alpha, \beta \in \text{Sper } A$, a connected set in $\text{Sper } A$ having certain properties.

Secondly, we construct a class of connected subsets of $\text{Sper } A$ which, though not in itself enough for the proof of the Pierce–Birkhoff conjecture, is the first and simplest example of the sort of connected sets we really need, and which constitutes the first step in our program for a proof of the Pierce–Birkhoff conjecture in dimension greater than 2. The precise relation of the main connectedness theorem of the present paper (Theorem 1.1) to the Pierce–Birkhoff conjecture, that is, the part of the program which is left for the subsequent papers, is explained in more detail later in this Introduction. In the forthcoming paper [12], the next paper of the series, we plan to prove the separation conjecture for $n = 3$ by constructing higher level connected sets in the real spectrum, based on the prototype constructed in the present paper.

Thirdly, we give two proofs of the Connectedness conjecture, one elementary and one invoking the main connectedness theorem of this paper, in the special case when one of α and β is monomial (this is explained in more detail later in this Introduction).

The rest of this paper is organized as follows. In §3 we define the valuation ν_α of $B(\alpha)$ associated to a point α of the real spectrum. The valuation ν_α is defined using the order \leq_α . It has following properties:

- (1) $\nu_\alpha(B[\alpha]) \geq 0$
- (2) If B is an R -algebra then for any positive elements $y, z \in B(\alpha)$,

$$\nu_\alpha(y) < \nu_\alpha(z) \implies y > Nz, \forall N \in R. \quad (2)$$

We also explain the geometric interpretation of a point of the real spectrum as a semi-curve.

Notation: To simplify the notation, we will write x in place of (x_1, \dots, x_n) .

For a point α of the real spectrum, let Γ_α denote the value group of ν_α . We say that α is **monomial** if it is given by a semi-curve of the form $t \rightarrow (c_1 t^{a_1}, \dots, c_n t^{a_n})$, where $c_j \in R$ and the a_j are strictly positive elements, \mathbb{Q} -linearly independent elements of Γ_α . Note that if α is monomial then

$$k_\alpha = R \quad (3)$$

Take a point $\alpha \in \text{Sper } A$. For an element $a \in \Gamma_\alpha$, let us denote the ν_α -ideal of value a and P_{a+} the greatest ν_α -ideal properly contained in P_a . In other words, we have

$$P_a = \{y \in A \mid \nu(y) \geq a\}, \quad (4)$$

$$P_a = \{y \in A \mid \nu(y) > a\}. \quad (5)$$

As an application of the above ideas, we give in §4 a proof of the Connectedness conjecture (and hence also of the Pierce–Birkhoff conjecture the abstract formulation) in the special case when for all the ν_α -ideals containing $< \alpha, \beta >$ we have

$$\dim_R \frac{P_a}{P_{a+}} = 1 \quad (6)$$

(as an R -vector space) and P_a is generated by monomials in x . In particular, this proves the Connectedness and the Pierce–Birkhoff conjectures in the case when one of α and β is a monomial point of the real spectrum.

The proof goes as follows. Let the notation be as in the statement of the Connectedness conjecture. By assumption, $g_i \notin < \alpha, \beta >$ so $\nu_\alpha(g_i) < \nu_\alpha(< \alpha, \beta >)$. The assumption (6) and the monomiality of P_a for $a < \nu_\alpha(< \alpha, \beta >)$ implies that the R -vector space $\frac{P_a}{P_{a+}}$ is generated by a natural image of a monomial in x . Hence, for $1 \leq i \leq s$, g_i can be written as the sum of a dominant monomial $M(i)$, of value $\nu_\alpha(g_i) < \nu_\alpha(< \alpha, \beta >)$, plus an R -linear combination of monomials of value strictly greater than $\nu_\alpha(g_i)$. Let us denote $M(i)$ by $c_i x^{\gamma(i)}$, with $\gamma(i) \in \mathbb{N}^n$, $c_i \in R$, and the non-dominant monomials appearing in g_i by $c_{ji} x^{\gamma_{ji}}$, $c_{ji} \in R$, $\gamma_{ji} \in \mathbb{N}^n$, $1 \leq j \leq N_i$:

$$g_i = c_i x^{\gamma(i)} + \sum_{j=1}^{N_i} c_{ji} x^{\gamma_{ji}},$$

where

$$\nu_\alpha(x^{\gamma(i)}) < \nu_\alpha(x^{\gamma_{ji}}), \quad 1 \leq j \leq N_i. \quad (7)$$

Now, Proposition 1.4 (2) and the fact that $\nu_\alpha(x^{\gamma(i)}) < \nu_\alpha(< \alpha, \beta >)$ imply that

$$\nu_\beta(x^{\gamma(i)}) < \nu_\beta(x^{\gamma_{ji}}), \quad 1 \leq j \leq N_i. \quad (8)$$

By definition of $< \alpha, \beta >$, if $x_q \notin < \alpha, \beta >$ then the sign of $x_q(\alpha)$ is the same as that of $x_q(\beta)$. Next, (2), (7) and (8) imply that

$$\left| x^{\gamma(i)} \right| \geq_\alpha N |x^{\gamma_{ji}}| \quad \text{for all } N \in R \quad \text{and} \quad 1 \leq j \leq N_i, \quad (9)$$

and similarly for $>_\beta$. Let C denote the set of all the points $\delta \in \text{Sper } A$ such that

$$\left| c_i x^{\gamma(i)} \right| \geq_\delta N_i |c_{ji} x^{\gamma_{ji}}| \quad \text{for} \quad 1 \leq j \leq N_i, \quad (10)$$

and $x_q(\delta)$ has the same sign as $x_q(\alpha)$ for all q such that $x_q \notin < \alpha, \beta >$. Then $\alpha, \beta \in C$ and all of the g_i have constant sign on C , $1 \leq i \leq s$. In §4 we give an elementary proof of the following fact:

Lemma 1.1 *Fix an index $l \in \{0, \dots, n\}$. Let C denote the subset of $\text{Sper } A$ defined by specifying $\text{sgn } x_q$ (which can be either strictly positive on all of C or strictly negative on all of C) for $q \in \{1, \dots, l\}$ and by imposing, in addition, finitely many monomial inequalities of the form*

$$\left| d_i x^{\lambda_i} \right| \geq \left| x^{\theta_i} \right|, \quad 1 \leq i \leq M \quad (11)$$

where $d_i \in R \setminus \{0\}$, $\lambda_i, \theta_i \in \mathbb{N}^n$ and x_q may appear only on the right hand side of the inequalities (38) for $q \in \{l+1, \dots, n\}$. Then C is connected.

This completes the proof of the Connectedness conjecture in this special case. Another connected set C which can do the same job is the set of all δ such that the inequalities (7) are satisfied with α replaced by δ and $x_q(\delta)$ has the same sign as $x_q(\alpha)$ for all q such that $x_q \notin \langle \alpha, \beta \rangle$. The connectedness of such a set C is an immediate corollary of the main connectedness theorem of the present paper — Theorem 1.1, stated below. This will be explained in more detail at the end of the paper.

In §5 we study the behaviour of certain subsets of the real spectrum under blowing up.

In §6 we recall and adapt to our context some known results on resolution of singularities, of a purely combinatorial nature. These results can be considered as a special case of the desingularization of toric varieties or Hironaka's game [20]. Since they are easy to prove, we chose to include complete proofs. The conclusion of this section is an algorithm for resolving singularities of any binomial by iterating combinatorial (toric) blowings up along non-singular centers.

Finally, §7 is devoted to the proof of the main theorem. Let $A = R[x_1, \dots, x_n]$ be a polynomial ring and let $\omega_{ij}, \theta_{il} \in \mathbb{Q}, i \in \{1, \dots, n\}, j \in \{1, \dots, q\}, l \in \{1, \dots, u\}$, where q and u are positive integers.

Let ν_δ be the valuation associated to the point $\delta \in \text{Sper}(A)$, defined in §3.

Notation: We will write $\nu_\delta(x)$ for the n -tuple $(\nu_\delta(x_1), \dots, \nu_\delta(x_n))$.

Let

$$h_j(\nu_\delta(x)) = \sum_{i=1}^n \omega_{ij} \nu_\delta(x_i) \text{ for } j \in \{1, \dots, q\},$$

and

$$z_l(\nu_\delta(x)) = \sum_{i=1}^n \theta_{il} \nu_\delta(x_i) \text{ for } l \in \{1, \dots, u\}.$$

Theorem 1.1 *The set*

$$S = \{ \delta \in \text{Sper}(A) \mid x_i >_\delta 0, i \in \{1, \dots, n\}, h_j(\nu_\delta(x)) > 0, j \in \{1, \dots, q\}, \\ z_l(\nu_\delta(x)) = 0, l \in \{1, \dots, u\} \} \quad (12)$$

is connected in the spectral topology.

In other words, subsets of $\text{Sper } A$ defined by finitely many \mathbb{Q} -linear equations and *strict* inequalities on $\nu(x_1), \dots, \nu(x_n)$ are connected.

To prove Theorem 1.1, we first reduce it to the following Proposition:

Proposition 1.6 *Assume that $u = 0$ in (12), that is, S is defined by finitely many strict \mathbb{Q} -linear inequalities and no equalities. Then S is connected.*

While revising this paper, we found a very simple proof of Proposition 1.6 using Lemma 1.1 (the first proof of Proposition 1.6 in the current version). However, we decided to keep our original proof because we feel that it sheds some light on the geometry of the real spectrum. Roughly speaking, this proof (now called the second proof of Proposition 1.6) can be thought of as a form of “path connectedness” of the set S . Along the way towards the second proof, we exhibit two other types of connected sets in $\text{Sper } A$, whose connectedness we believe to be of interest in its own right.

In the forthcoming paper [12], we will develop the theory of approximate roots of a valuation. Given a ring A and a valuation ν , non-negative on A , a family of approximate roots is a collection $\{Q_i\}$, finite or countable, of elements of A , having some additional properties as outlined below. A **generalized monomial** (with respect to a given collection $\{Q_i\}$ of approximate roots) is, by definition, an element of A of the form $\prod_j Q_j^{\gamma_j}$, $\gamma_j \in \mathbb{N}$. The main defining properties of the approximate roots are the fact that every ν -ideal I in A is generated by the generalized monomials contained in it, that is, generalized monomials $\prod_j Q_j^{\gamma_j}$ satisfying

$$\sum_j \gamma_j \nu(Q_j) \geq \nu(I),$$

and the fact that for each i , Q_i is described by an explicit formula in terms of Q_1, \dots, Q_{i-1} . In particular, the valuation ν is completely determined by the set $\{Q_i\}$ and the values $\nu(Q_i)$.

We then show that every element $g \in A$ can be written as a finite sum of the form

$$g = c\mathbf{Q}^\theta + \sum_{j=1}^N c_j \mathbf{Q}^{\delta_j}, \quad (13)$$

where c and c_j are elements of A such that $\nu(c) = \nu(c_j) = 0$ and \mathbf{Q}^θ and \mathbf{Q}^{δ_j} are generalized monomials such that

$$\nu(\mathbf{Q}^\theta) < \nu(\mathbf{Q}^{\delta_j}) \text{ for } 1 \leq j \leq N. \quad (14)$$

Now let $\delta \in \text{Sper}(A)$ and let $\nu = \nu_\delta$. Then, by (2) and (14) the sign of g with respect to \leq_δ is determined by the sign of its leading coefficient c .

Let α and β be two points of $\text{Sper}(A)$ having a common specialization. In [12] we will give an explicit description of the set of generalized monomials (with respect to approximate roots Q_i common to ν_α and ν_β) which generate the separating ideal $\langle \alpha, \beta \rangle$. Furthermore, we show that all the approximate roots Q_i for ν_α such that $Q_i \notin \langle \alpha, \beta \rangle$ are also approximate roots for ν_β and vice versa.

Now let g be as in the separation conjecture. The fact that $g \notin \langle \alpha, \beta \rangle$ implies the existence of an expression (13) in which all the approximate roots Q_i are common for ν_α and ν_β and the inequalities (14) hold for both $\nu = \nu_\alpha$ and $\nu = \nu_\beta$. The inequalities (14) can be viewed as linear inequalities on $\nu(Q_1), \dots, \nu(Q_t)$ with integer coefficients.

To prove the separation conjecture, we look for connected sets $C \subset \text{Sper}(A)$ having the following properties:

- (1) $\alpha, \beta \in C$;
- (2) the Q_i appearing in (13) are approximate roots simultaneously for all the ν_δ , $\delta \in C$;
- (3) the inequalities (14) hold for $\nu = \nu_\delta$, for all $\delta \in C$;
- (4) the leading coefficient c has constant sign on C .

Once such a C is found, (2) and (14) imply that the sign of g on C is constant, which proves that α and β lie in the same connected component of $\text{Sper}(A) \setminus \{g = 0\}$.

In [12] we will define the set

$$C(g, \alpha, \beta) = \left\{ \delta \in \text{Sper}(A) \mid \begin{array}{l} Q_i > 0, i \in \{1, \dots, t\}, \sum_{i=1}^t \omega_{ij} \nu_\delta(Q_i) > 0, j \in \{1, \dots, q\}, \\ \sum_{i=1}^t \lambda_{ij} \nu_\delta(Q_i) = 0, j \in \{1, \dots, l\} \end{array} \right\}, \quad (15)$$

where $\omega_{ij}, \lambda_{ij} \in \mathbb{Z}$, all the inequalities (14) described above appear among the $\sum_{i=1}^t \omega_{ij} \nu_\delta(Q_i) > 0$ and the remaining equalities and inequalities on the right hand side of (15) encode the fact that Q_1, \dots, Q_t are approximate roots for ν_δ for all δ belonging to $C(g, \alpha, \beta)$. If g_1, \dots, g_s is a finite collection of elements of $A \setminus \langle \alpha, \beta \rangle$, we put

$$C(g_1, \dots, g_s, \alpha, \beta) = \bigcap_{i=1}^s C(g_i, \alpha, \beta).$$

By construction, both sets $C(g, \alpha, \beta)$ and $C(g_1, \dots, g_s, \alpha, \beta)$ contain α and β , the element g does not change sign on $C(g, \alpha, \beta)$ and none of the elements g_1, \dots, g_s change sign on $C(g_1, \dots, g_s, \alpha, \beta)$. Thus to prove the separation conjecture it is sufficient to prove that $C(g, \alpha, \beta)$ is connected and to prove the Pierce–Birkhoff conjecture it is sufficient to prove that $C(g_1, \dots, g_s, \alpha, \beta)$ is connected.

A proof of the connectedness of $C(g, \alpha, \beta)$ and $C(g_1, \dots, g_s, \alpha, \beta)$ is relegated to future papers. In any case, the most delicate part of the proof of the separation (resp. the Pierce–Birkhoff) conjecture is proving the connectedness of $C(g, \alpha, \beta)$ (resp. $C(g_1, \dots, g_s, \alpha, \beta)$). The connectedness theorem of the present paper is the special case of the desired result in which the finite set $\{Q_1, \dots, Q_t\}$ is a subset of the set of variables $\{x_1, \dots, x_n\}$. In [12] we plan to reduce the connectedness of $C(g, \alpha, \beta)$ to this special case using sequences of blowings up of the form described in §6: we will construct a sequence π of blowings up such that the total transform of each Q_i , $i \in \{1, \dots, t\}$, is a usual monomial with respect to the new coordinates, times a unit. The preimage of $C(g, \alpha, \beta)$ under π has the form (12) in the new coordinates. This will reduce the connectedness of $C(g, \alpha, \beta)$ to that of sets of the form S , proved in this paper.

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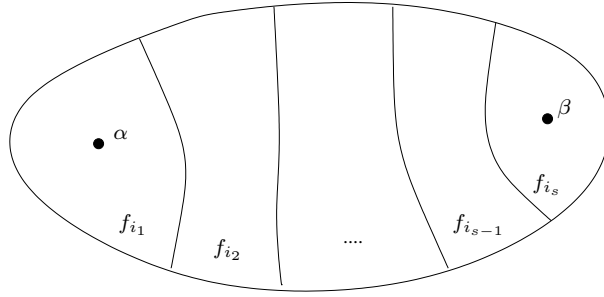
We also thank the referee for a very thorough reading of the paper and for his constructive criticism which helped us improve the exposition and eliminate numerous inaccuracies.

2 The Connectedness conjecture implies the Pierce–Birkhoff conjecture

Theorem 2.1 *The Connectedness conjecture implies the Pierce–Birkhoff conjecture.*

Proof: We will assume the Connectedness conjecture and deduce the Pierce–Birkhoff conjecture in the form of Conjecture 2. Let $f \in PW(A)$ and let $\{f_i\}_{1 \leq i \leq r}$ denote the elements of A which represent f on the various closed constructible subsets $P_i \subset \text{Sper } A$. Let $\alpha, \beta \in \text{Sper } A$ and let

$$T = \{\{i, j\} \subset \{1, \dots, r\} \mid f_i - f_j \notin \langle \alpha, \beta \rangle\}. \quad (16)$$



We apply the Connectedness conjecture to the finite collection $\{f_i - f_j \mid \{i, j\} \in T\}$ of elements of A . By the Connectedness conjecture, there exists a connected subset $C \subset \text{Sper } A$ such that $\alpha, \beta \in C$ and

$$C \cap \{f_i - f_j = 0\} = \emptyset \text{ for all } \{i, j\} \in T. \quad (17)$$

Let $J \subset \{1, \dots, r\}$ be the set of all the indices j having the following property: there exist $i_1, \dots, i_s \in \{1, \dots, r\}$ such that

$$\alpha = i_1, \quad (18)$$

$$j = i_s \quad (19)$$

and for each $q \in \{1, \dots, s-1\}$, we have

$$C \cap \{f_{i_q} - f_{i_{q+1}} = 0\} \neq \emptyset. \quad (20)$$

Let $F = \bigcup_{j \in J} (P_j \cap C)$. We have $\alpha \in F$ by definition.

Claim: $F = C$; in particular, $\beta \in F$.

Proof of Claim. Let $J^c = \{1, \dots, r\} \setminus J$ and $G = \bigcup_{j \in J^c} (P_j \cap C)$. Clearly

$$C = F \cup G \quad (21)$$

and both sets F and G are closed in the induced topology of C (since all the P_j are closed). Next, we show that

$$F \cap G = \emptyset \quad (22)$$

Indeed, if $\delta \in F \cap G$ then there exist $j \in J$ and $j' \in J^c$ such that $\delta \in P_j \cap P_{j'}$. But then $f_j(\delta) = f_{j'}(\delta)$, so $\delta \in C \cap \{f_j - f_{j'} = 0\}$, hence $j' \in J$, a contradiction. This proves (22), so the union in (21) is a disjoint union.

Now, since C is connected and $F \neq \emptyset$ (since $\alpha \in F$), the expression (21) of C as a disjoint union of closed sets implies that $G = \emptyset$. Hence $\beta \in F$, which completes the proof of the Claim.

Let $j \in J$ be such that $\beta \in P_j$, so that $f_j = f_\beta$. Let i_1, \dots, i_s be as in (18)–(20), expressing the fact that $j \in J$. Together, (16), (17) and (20) imply that $f_{i_q} - f_{i_{q+1}} \in \langle \alpha, \beta \rangle$ for all $q \in \{1, \dots, s-1\}$. In view of (18) and (19), we obtain $f_\alpha - f_\beta \in \langle \alpha, \beta \rangle$, as desired. \square

3 The valuation associated to a point in the real spectrum

Let B be a ring and α a point in $\text{Sper } B$. In this section we define the valuation ν_α of $B(\alpha)$, associated to α . We also give a geometric interpretation of points in $\text{Sper } B$ as semi-curvettes.

Terminology: If B is an integral domain, the phrase “valuation of B ” will mean “a valuation of the field of fractions of B , non-negative on B ”. Also, we will sometimes commit the following abuse of notation. Given a ring B , a prime ideal $\mathfrak{p} \subset B$, a valuation ν of $\frac{B}{\mathfrak{p}}$ and an element $x \in B$, we will write $\nu(x)$ instead of $\nu(x \bmod \mathfrak{p})$, with the usual convention that $\nu(0) = \infty$, which is taken to be greater than any element of the value group.

First, we define the valuation ring R_α by

$$R_\alpha = \{x \in B(\alpha) \mid \exists z \in B[\alpha], |x| \leq_\alpha z\}.$$

That R_α is, in fact, a valuation ring, follows because for any $x \in B(\alpha)$, either $x \in R_\alpha$ or $\frac{1}{x} \in R_\alpha$. The maximal ideal of R_α is $M_\alpha = \left\{x \in B(\alpha) \mid |x| < \frac{1}{|z|}, \forall z \in B[\alpha] \setminus \{0\}\right\}$; its residue field k_α comes equipped with a total ordering, induced by \leq_α . For a ring B let $U(B)$ denote the multiplicative group of units of B . Recall that $\Gamma_\alpha \cong \frac{B(\alpha) \setminus \{0\}}{U(R_\alpha)}$ and that the valuation ν_α can be identified with the natural homomorphism

$$B(\alpha) \setminus \{0\} \rightarrow \frac{B(\alpha) \setminus \{0\}}{U(R_\alpha)}.$$

By definition, we have a natural ring homomorphism

$$B \rightarrow R_\alpha \tag{23}$$

whose kernel is \mathfrak{p}_α .

Remark 3.1 ([3], [11], [5] 10.1.10, p. 217) *Conversely, the point α can be reconstructed from the ring R_α by specifying a certain number of sign conditions (finitely many conditions when B is noetherian), as we now explain. Take a prime ideal $\mathfrak{p} \subset B$ and a valuation ν of $\kappa(\mathfrak{p}) := \frac{B_\mathfrak{p}}{\mathfrak{p}B_\mathfrak{p}}$, with value group Γ . Let*

$$r = \dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$$

(if B is not noetherian, it may happen that $r = \infty$). Let x_1, \dots, x_r be elements of $\kappa(\mathfrak{p})$ such that $\nu(x_1), \dots, \nu(x_r)$ induce a basis of the \mathbb{F}_2 -vector space $\Gamma/2\Gamma$. Then for every $x \in \kappa(\mathfrak{p})$, there exists $f \in \kappa(\mathfrak{p})$, and a unit u of R_ν such that $x = ux_1^{\epsilon_1} \cdots x_r^{\epsilon_r} f^2$ with $\epsilon_i \in \{0, 1\}$ (to see this, note that for a suitable choice of f and ϵ_j the value of the quotient u of x by the product $x_1^{\epsilon_1} \cdots x_r^{\epsilon_r} f^2$ is 0, hence u is invertible in R_ν). Now, specifying a point $\alpha \in \text{Sper } B$ supported at \mathfrak{p} amounts to specifying a valuation ν of $\frac{B}{\mathfrak{p}}$, a total ordering of the residue field k_ν of R_ν , and the sign data $\text{sgn } x_1, \dots, \text{sgn } x_r$. For $x \notin \mathfrak{p}$, the sign of x is given by the product $\text{sgn}(x_1)^{\epsilon_1} \cdots \text{sgn}(x_r)^{\epsilon_r} \text{sgn}(u)$, where $\text{sgn}(u)$ is determined by the ordering of k_ν .

Points of $\text{Sper } B$ admit the following geometric interpretation (we refer the reader to [7], [9], [17], p. 89 and [18] for the construction and properties of generalized power series rings and fields).

Definition 3.1 *Let k be a field and Γ an ordered abelian group. The generalized formal power series ring $k[[t^\Gamma]]$ is the ring formed by elements of the form $\sum_\gamma a_\gamma t^\gamma$, $a_\gamma \in k$ such that the set $\{\gamma \mid a_\gamma \neq 0\}$ is well ordered.*

The ring $k[[t^\Gamma]]$ is equipped with the natural t -adic valuation v with values in Γ , defined by $v(f) = \inf\{\gamma \mid a_\gamma \neq 0\}$ for $f = \sum_\gamma a_\gamma t^\gamma \in k[[t^\Gamma]]$. Specifying a total ordering on k and

$\dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$ sign conditions defines a total ordering on $k[[t^\Gamma]]$. In this ordering $|t|$ is smaller than any positive element of k . For example, if $t^\gamma > 0$ for all $\gamma \in \Gamma$ then $f > 0$ if and only if $a_{v(f)} > 0$.

For an ordered field k , let \bar{k} denote the real closure of k . The following result is a variation on a theorem of Kaplansky ([9], [10]) for valued fields equipped with a total ordering.

Theorem 3.1 ([18], p. 62, Satz 21) *Let K be a real valued field, with residue field k and value group Γ . There exists an injection $K \hookrightarrow \bar{k}((t^\Gamma))$ of real valued fields.*

Let $\alpha \in \text{Sper } B$ and let Γ be the value group of ν_α . In view of (23) and the Remark above, specifying a point $\alpha \in \text{Sper } B$ is equivalent to specifying a total order of k_α , a morphism

$$B[\alpha] \rightarrow \bar{k}_\alpha[[t^\Gamma]]$$

and $\dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$ sign conditions as above.

We may pass to usual spectra to obtain morphisms

$$\text{Spec } (\bar{k}_\alpha[[t^\Gamma]]) \rightarrow \text{Spec } B[\alpha] \rightarrow \text{Spec } B.$$

In particular, if $\Gamma = \mathbb{Z}$, we obtain a **formal curve** in $\text{Spec } B$ (an analytic curve if the series are convergent). This motivates the following definition:

Definition 3.2 *Let k be an ordered field. A k -curvette on $\text{Sper}(B)$ is a morphism of the form*

$$\alpha : B \rightarrow k[[t^\Gamma]],$$

where Γ is an ordered group. A k -**semi-curvette** is a k -curvette α together with a choice of the sign data $\text{sgn } x_1, \dots, \text{sgn } x_r$, where x_1, \dots, x_r are elements of B whose t -adic values induce an \mathbb{F}_2 -basis of $\Gamma/2\Gamma$.

We have thus explained how to associate to a point α of $\text{Sper } B$ a \bar{k}_α -semi-curvette. Conversely, given an ordered field k , a k -semi-curvette α determines a prime ideal \mathfrak{p}_α (the ideal of all the elements of B which vanish identically on α) and a total ordering on B/\mathfrak{p}_α induced by the ordering of the ring $k[[t^\Gamma]]$ of formal power series. These two operations are inverse to each other. This establishes a one-to-one correspondence between semi-curvettes and points of $\text{Sper } B$.

Below, we will often describe points in the real spectrum by specifying the corresponding semi-curvettes.

Example: Consider the curvette $\mathbb{R}[x, y] \rightarrow \mathbb{R}[[t]]$ defined by $x \mapsto t^2$, $y \mapsto t^3$, and the semi-curvette given by declaring, in addition, that t is positive. This semi-curvette is nothing but the upper branch of the cusp.

Throughout this paper, we study sets of points of the real spectrum defined by certain properties of their associated valuations. As the point δ varies, so does ν_δ and its value group Γ_δ . In order to describe such sets in the real spectrum, we need to embed Γ_δ in some “universal” ordered group.

Notation and convention: Let us denote by Γ the ordered group \mathbb{R}_{lex}^n . This means that elements of Γ are compared as words in a dictionary: we say that $(a_1, \dots, a_n) < (a'_1, \dots, a'_n)$ if and only if there exists $j \in \{1, \dots, n\}$ such that $a_q = a'_q$ for all $q < j$ and $a_j < a'_j$.

The reason for introducing Γ is that by Abhyankar’s inequality we have $\text{rank } \nu_\delta \leq \dim A = n$ for all $\delta \in \text{Sper } A$, so the value group Γ_δ can be embedded into Γ as an ordered subgroup (of

course, this embedding is far from being unique). Let Γ_+ be the semigroup of non-negative elements of Γ .

Fix a strictly positive integer ℓ . In order to deal rigourously with ℓ -tuples of elements of Γ_δ despite the non-uniqueness of the embedding $\Gamma_\delta \subset \Gamma$, we introduce the category $\mathcal{OGM}(\ell)$, as follows. An object in $\mathcal{OGM}(\ell)$ is an ordered abelian group G together with ℓ fixed generators a_1, \dots, a_ℓ (such an object will be denoted by (G, a_1, \dots, a_ℓ)). A morphism from (G, a_1, \dots, a_ℓ) to $(G', a'_1, \dots, a'_\ell)$ is a homomorphism $G \rightarrow G'$ of ordered groups which maps a_j to a'_j for each j .

Given $(G, a_1, \dots, a_\ell), (G', a'_1, \dots, a'_\ell) \in \text{Ob}(\mathcal{OGM}(\ell))$, the notation

$$(a_1, \dots, a_\ell) \underset{\circ}{\sim} (a'_1, \dots, a'_\ell) \quad (24)$$

will mean that (G, a_1, \dots, a_ℓ) and $(G', a'_1, \dots, a'_\ell)$ are isomorphic in $\mathcal{OGM}(\ell)$.

Take an element

$$a = (a_1, \dots, a_\ell) \in \Gamma_+^\ell.$$

Let $G \subset \Gamma$ be the ordered group generated by a_1, \dots, a_ℓ . Then $(G, a_1, \dots, a_\ell) \in \text{Ob}(\mathcal{OGM}(\ell))$. For each $\delta \in \text{Sper}(A)$, let Γ_δ denote the value group of the associated valuation ν_δ and Γ_δ^* the subgroup of Γ_δ generated by $\nu_\delta(x_1), \dots, \nu_\delta(x_n)$. In this way, we associate to δ an object $(\Gamma_\delta^*, \nu_\delta(x_1), \dots, \nu_\delta(x_n)) \in \text{Ob}(\mathcal{OGM}(n))$.

4 A proof of the Connectedness conjecture in the case when one of α and β is monomial

Consider two points $\alpha, \beta \in \text{Sper } A$. For an element $a \in \Gamma_\alpha$, let P_a denote the ν_α -ideal of value a and P_{a+} the greatest ν_α -ideal which is properly contained in P_a (cf. (4)–(5)). The purpose of this section is to prove the Connectedness (and hence also the Pierce–Birkhoff) conjecture assuming that P_a is generated by monomials in x whenever $a \leq \nu_\alpha(< \alpha, \beta >)$ and that

$$\dim_R \frac{P_a}{P_{a+}} = 1 \quad \text{whenever } a < \nu_\alpha(< \alpha, \beta >). \quad (25)$$

We start with a general (and known) result on semi-algebraic connectedness of constructible sets in the Euclidean space. We give a proof since we did not find the exact statement we needed in the literature (a slightly weaker form of it is given in [2], p. 52, Proposition–Definition 62). We state and prove the result in greater generality than needed in this section, because of the applications we have in mind later in the paper.

Consider the totally ordered set $R_\infty := \{-\infty\} \cup R \cup \{+\infty\}$, equipped with the order topology. The map

$$\phi(x) = -\frac{1}{1+x} + 1 \quad \text{if } x \geq 0 \quad (26)$$

$$= \frac{1}{1-x} - 1 \quad \text{if } x \leq 0 \quad (27)$$

defines a homeomorphism from R_∞ to the interval $[-1, 1]$. Let $1 \leq s \leq n-1$. The homeomorphism ϕ gives rise to a homeomorphism

$$\phi^{n-s} : R_\infty^{n-s} \rightarrow [-1, 1]^{n-s}$$

in the obvious way, that is, by applying the map ϕ separately to each component. A **closed semi-algebraic** subset of $R^s \times R_\infty^{n-s}$ is, by definition, the preimage of a (relatively) closed semi-algebraic subset of $R^s \times [-1, 1]^{n-s}$ with respect to the product topology of $R^s \times [-1, 1]^{n-s}$, under the map (id, ϕ^{n-s}) . Let G be a semi-algebraic subset of R^s . A mapping

$$G \rightarrow R_\infty^{n-s}$$

is said to be **semi-algebraic** if its graph is a closed semi-algebraic set.

Take a semi-algebraic set $G \subset R^s$. Fix two continuous semi-algebraic mappings

$$f, g : G \rightarrow R_\infty^{n-s},$$

having the following property. Write $f = (f_{j+1}, \dots, f_n)$, $g = (g_{j+1}, \dots, g_n)$, where f_j and g_j are continuous semi-algebraic R_∞ -valued functions on G . We require, for each $(a_1, \dots, a_s) \in R^s$ and each $j \in \{s+1, \dots, n\}$, the inequality

$$f_j(a_1, \dots, a_s) \leq g_j(a_1, \dots, a_s).$$

Let $D_{f,g}$ denote the semi-algebraic subset of $R^s \times R_\infty^{n-s}$ defined by

$$D_{f,g} = \{(a_1, \dots, a_n) \in R^n \mid (a_1, \dots, a_s) \in G, f_j(a_1, \dots, a_s) \leq a_j \leq g_j(a_1, \dots, a_s), s < j \leq n\}. \quad (28)$$

Let $p : R^s \times R_\infty^{n-s} \rightarrow R^s$ be the canonical projection and

$$p_{f,g} : D_{f,g} \rightarrow G \quad (29)$$

the restriction of p to $D_{f,g}$.

Proposition 4.1 *Assume that G is semi-algebraically connected. Then so is the semi-algebraic set $D_{f,g}$.*

Proof: Composing both f and g with the continuous semi-algebraic mapping ϕ^{n-s} , we may assume that the images of f and g lie in R_∞^{n-s} rather than in R_∞^{n-s} .

The map $p_{f,g}$ has semi-algebraically connected fibers (in fact, each fiber of $p_{f,g}$ is the parallelepiped defined by the inequalities

$$f_j(a_1, \dots, a_s) \leq x_j \leq g_j(a_1, \dots, a_s), \quad j \in \{s+1, \dots, n\},$$

which is semi-algebraically connected ([5], Proposition 2.4.3, p. 31)). Suppose $D_{f,g}$ were not semi-algebraically connected. Then there would exist open semi-algebraic sets $U, V \subset R^n$ such that

$$D_{f,g} = (U \cap D_{f,g}) \amalg (V \cap D_{f,g}),$$

with $U \cap D_{f,g} \neq \emptyset$, $V \cap D_{f,g} \neq \emptyset$. Let $U_0 = p_{f,g}(U \cap D_{f,g})$, $V_0 = p_{f,g}(V \cap D_{f,g})$.

Claim: We have $U_0 \cap V_0 = \emptyset$.

Proof of the Claim: We prove the Claim by contradiction. Take a point $\xi \in U_0 \cap V_0$. Then $p_{f,g}^{-1}(\xi) = (p_{f,g}^{-1}(\xi) \cap U) \amalg (p_{f,g}^{-1}(\xi) \cap V)$. We have expressed the fiber $p_{f,g}^{-1}(\xi)$ as a disjoint union of two non-empty, relatively open semi-algebraic sets which contradicts the semi-algebraic connectedness of $p_{f,g}^{-1}(\xi)$. The Claim is proved.

It follows from the Claim that

$$U \cap D_{f,g} = p_{f,g}^{-1}(U_0) \quad \text{and} \quad (30)$$

$$V \cap D_{f,g} = p_{f,g}^{-1}(V_0). \quad (31)$$

We have

$$G = U_0 \coprod V_0. \quad (32)$$

Moreover, (30)–(31) imply that U_0 is the preimage of the open semi-algebraic set U under the continuous semi-algebraic mapping

$$\left(id, \frac{f+g}{2} \right) : G \rightarrow R^n = R^s \times R^{n-s},$$

hence it is relatively open in G , and similarly for V_0 . Thus (32) expresses G as a disjoint union of non-empty relatively open semi-algebraic sets. Then G is not semi-algebraically connected, which gives the desired contradiction. \square

Remark 4.1 *Proposition 4.1 holds just as well if we replace one or both non-strict inequalities in (28) by strict ones (the same proof applies verbatim).*

Theorem 4.1 *Let the notation and assumptions be as in the beginning of this section. Take a finite collection g_1, \dots, g_s of elements of $A \setminus \langle \alpha, \beta \rangle$. Then there exists a connected set $C \subset \text{Sper } A$ such that $\alpha, \beta \in C$ and $C \cap \{g_i = 0\} = \emptyset$ for $i \in \{1, \dots, s\}$ (in other words, α and β belong to the same connected component of the set $\text{Sper } A \setminus \{g_1 \dots g_s = 0\}$).*

Remark 4.2 *Of course, the conclusion of Theorem 4.1 remains valid if its hypotheses are satisfied after applying an R -automorphism of A .*

Corollary 4.1 *Assume that at least one of α, β is monomial. Then the conclusion of the Connectedness conjecture (and hence also that of the Pierce–Birkhoff conjecture in the abstract formulation) holds for the pair α, β .*

Proof of Corollary 4.1 assuming Theorem 4.1: In general, for each $a \in \Gamma_\alpha$ we have a non-canonical injection

$$\frac{P_a}{P_{a+}} \subset k_\alpha \quad (33)$$

(of $\frac{A}{P_{0+}}$ -vector spaces), defined by picking an element $y \in P_a \setminus P_{a+}$ and for each $z \in P_a$ sending $z \bmod P_{a+}$ to the natural image of $\frac{z}{y}$ in k_α . Without loss of generality, we may assume that α is monomial. Then (3), (33) and the fact that R is a subfield of $\frac{A}{P_{0+}}$ imply (25) for all $a \in \Gamma_\alpha$, in particular, for $a < \nu_\alpha(\langle \alpha, \beta \rangle)$. Also, it is easy to see from the definition of monomial points of the real spectrum that the ideal P_a is monomial for all $a \in \Gamma_\alpha$, in particular for $a \leq \nu_\alpha(\langle \alpha, \beta \rangle)$. Thus the hypotheses of Theorem 4.1 are satisfied whenever α is monomial. This proves the Corollary. \square

Proof of Theorem 4.1: We have $g_i \notin \langle \alpha, \beta \rangle$ by assumption, so $\nu_\alpha(g_i) < \nu_\alpha(\langle \alpha, \beta \rangle)$ (since $\langle \alpha, \beta \rangle$ is a ν_α -ideal by Proposition 1.4 (1)). The assumption (25) and the monomiality of P_a for $a \leq \nu_\alpha(\langle \alpha, \beta \rangle)$ imply that the R -vector space $\frac{P_a}{P_{a+}}$ is generated by a natural image of a monomial in x . Hence, for $1 \leq i \leq s$, g_i can be written as the sum of a dominant monomial (which we will denote by $c_i x^{\gamma(i)}$, $\gamma(i) \in \mathbb{N}^n$, $c_i \in R$), of value $\nu_\alpha(g_i) < \nu_\alpha(\langle \alpha, \beta \rangle)$, plus an R -linear combination of monomials of value strictly greater than $\nu_\alpha(g_i)$. Let us denote these non-dominant monomials appearing in g_i by $c_{ji} x^{\gamma_{ji}}$, $c_{ji} \in R$, $\gamma_{ji} \in \mathbb{N}^n$, $1 \leq j \leq N_i$:

$$g_i = c_i x^{\gamma(i)} + \sum_{j=1}^{N_i} c_{ji} x^{\gamma_{ji}},$$

where

$$\nu_\alpha(x^{\gamma(i)}) < \nu_\alpha(x^{\gamma_{ji}}), \quad 1 \leq j \leq N_i. \quad (34)$$

Proposition 1.4 (2) and the fact that $\nu_\alpha(x^{\gamma(i)}) < \nu_\alpha(<\alpha, \beta>)$ imply that

$$\nu_\beta(x^{\gamma(i)}) < \nu_\beta(x^{\gamma_{ji}}), \quad 1 \leq j \leq N_i. \quad (35)$$

Now, (2), (34) and (35) imply that

$$|x^{\gamma(i)}| \geq_\alpha N |x^{\gamma_{ji}}| \quad \text{for all } N \in R \quad \text{and} \quad 1 \leq j \leq N_i, \quad (36)$$

and similarly for \geq_β . By definition of the separating ideal, if $x_q \notin <\alpha, \beta>$ then the sign of $x_q(\alpha)$ is the same as that of $x_q(\beta)$. Renumbering the x_q , if necessary, we may assume that there exists $l \in \{0, \dots, n\}$ such that $x_1, \dots, x_l \notin <\alpha, \beta>$ and $x_{l+1}, \dots, x_n \in <\alpha, \beta>$. Then for $q \in \{l+1, \dots, n\}$ the variable x_q appears only on the right side of the inequalities (34), (35) and (36) (otherwise we would have $\nu_\alpha(x_q) < \nu_\alpha(<\alpha, \beta>)$, a contradiction).

Let C denote the set of all the points $\delta \in \text{Sper } A$ such that

$$|c_i x^{\gamma(i)}| \geq_\delta N_i |c_{ji} x^{\gamma_{ji}}| \quad \text{for} \quad 1 \leq j \leq N_i, \quad (37)$$

$x_q(\delta)$ has the same sign as $x_q(\alpha)$ for all q such that $x_q \notin <\alpha, \beta>$, and for $q \in \{l+1, \dots, n\}$ the variable x_q appears only on the right side of the inequalities (37). Then $\alpha, \beta \in C$ and all of the g_i have constant sign on C , $1 \leq i \leq s$. Thus to complete the proof Theorem 4.1, it remains to prove the following Lemma:

Lemma 4.1 *Let C denote the subset of $\text{Sper } A$ defined by specifying $\text{sgn } x_q$ (which can be either strictly positive on all of C or strictly negative on all of C) for $q \in \{1, \dots, l\}$ and by imposing, in addition, finitely many monomial inequalities of the form*

$$|d_i x^{\lambda_i}| \geq |x^{\theta_i}|, \quad 1 \leq i \leq M \quad (38)$$

where $d_i \in R \setminus \{0\}$, $\lambda_i, \theta_i \in \mathbb{Q}_+^n$ and x_q may appear only on the right hand side of the inequalities (38) for $q \in \{l+1, \dots, n\}$. Then C is connected.

Convention on rational exponents. It is convenient to allow the exponents in (38) to be rational. We ascribe the obvious meaning to such an inequality by raising both sides to a suitable integer power to make all the exponents integer.

Remark 4.3 *Of course, Lemma 4.1 holds just as well if we replace the non-strict inequalities in (38) by strict ones (the proof is the same). For the proof of Theorem 4.1, it makes no difference whether we use strict or non-strict inequalities: both work equally well. We state Lemma 4.1 as we do because it will be crucial in the first proof of our main connectedness theorem — Theorem 1.1 — where non-strict inequalities are essential because there we need the set C to be closed as well as connected.*

Proof of Lemma 4.1: We proceed by induction on n . If $n = 1$, all of the inequalities (38) amount to either $|x_1| \leq c$ or $|x_1| \geq d$ for some $c, d \in R$, with the latter type of inequality allowed only if $l = 1$, in which case the sign of x_1 on C is fixed. Thus C is either empty or an interval of one of the forms $\{c' \leq x_1 \leq d'\}$, $\{0 < x_1 \leq d'\}$ or $\{c' \leq x_1 < 0\}$, $c', d' \in R$, which are well known to be connected.

Next, assume that $n > 1$ and that the result is known for $n - 1$. Without loss of generality, we may assume that there exists $M_1 \in \{0, \dots, M\}$ such that the inequalities (38) involve x_n for $1 \leq i \leq M_1$ and do not involve x_n for $M_1 < i \leq M$. First, suppose

$$l = n. \quad (39)$$

Let $\bar{x} = (x_1, \dots, x_{n-1})$ and for each i , $1 \leq i \leq M_1$, rewrite the i -th inequality (38) in one of the forms

$$|x_n| \leq |d'_i \bar{x}^{\bar{\lambda}_i}| \quad \text{or} \quad (40)$$

$$|x_n| \geq |d'_i \bar{x}^{\bar{\lambda}_i}| \quad (41)$$

where $d'_i \in R$ and $\bar{\lambda}_i \in \mathbb{Q}^{n-1}$. Without loss of generality, we may assume that there exists $T \in \{0, \dots, M_1\}$ such that for $i \leq T$ the i -th inequality is of the form (40) while for $T < i \leq M_1$ the i -th inequality is of the form (41). Then the inequalities (38) for $1 \leq i \leq M_1$ can be rewritten as

$$\max_{T < i \leq M_1} \left\{ |d'_i \bar{x}^{\bar{\lambda}_i}| \right\} \leq |x_n| \leq \min_{1 \leq i \leq T} \left\{ |d'_i \bar{x}^{\bar{\lambda}_i}| \right\}. \quad (42)$$

Here the first inequality in (42) is taken to be non-existent if $T = M_1$ and the second inequality is considered non-existent if $T = 0$.

By (39), x_n is either strictly positive on all of C or strictly negative on all of C . Without loss of generality, we may assume that $x_n > 0$ on C . Then we may replace $|x_n|$ by x_n in the inequalities (42). Consider the set $\tilde{C}_{n-1} \subset \text{Sper } R[x_1, \dots, x_{n-1}]$, defined by the inequalities

$$|d'_i \bar{x}^{\bar{\lambda}_i}| \leq |d'_j \bar{x}^{\bar{\lambda}_j}|, \quad i \in \{T+1, \dots, M_1\}, j \in \{1, \dots, T\}, \quad (43)$$

all the inequalities (38) not involving x_n (that is, those with $M_1 < i \leq M$), and the same sign conditions on $\text{sgn } x_1, \dots, \text{sgn } x_{n-1}$ as appear in the definition of C . By the induction assumption, \tilde{C}_{n-1} is a connected, constructible set of $\text{Sper } R[x_1, \dots, x_{n-1}]$.

A subset of R^n defined by the conditions of the form $\bar{x} \in C_{n-1}$,

$$f(\bar{x}) \leq x_n \leq g(\bar{x}), \quad (44)$$

where f and g are two continuous, semi-algebraic functions defined on a connected, open semi-algebraic set $C_{n-1} \subset R^{n-1}$ with $f(\bar{x}) \leq g(\bar{x})$, is semi-algebraically connected by Proposition 4.1 (if $T = M_1$ in (42), we take $f = 0$ and the first inequality in (44) becomes strict). Hence C is connected by Proposition 1.2. This proves Lemma 4.1 in the case when $l = n$.

If $l < n$, the above argument also goes through, with the following changes:

(1) Since $l < n$, the inequalities of the form (41) do not appear in the definition of C , so that $T = M_1$.

(2) The inequality (42) becomes

$$|x_n| \leq \min_{1 \leq i \leq M_1} \left\{ |d'_i \bar{x}^{\bar{\lambda}_i}| \right\}. \quad (45)$$

Note that if x_q appears on the right hand side of (45) with a negative exponent then $q < l$ and so x_q is either strictly positive or strictly negative on all of C .

(3) The sets $\tilde{C}_{n-1} \subset \text{Sper } R[x_1, \dots, x_{n-1}]$ and $C_{n-1} \subset R^{n-1}$ are defined by all the inequalities (38) not involving x_n (that is, those with $M_1 < i \leq M$), and sign conditions on $\text{sgn } x_1, \dots, \text{sgn } x_{n-1}$ which appear in the definition of C .

(4) The functions f and g of (44) are given by

$$f = - \min_{1 \leq i \leq M_1} \left\{ \left| d'_i \bar{x}^{\bar{\lambda}_i} \right| \right\} \quad \text{and} \quad (46)$$

$$g = \min_{1 \leq i \leq M_1} \left\{ \left| d'_i \bar{x}^{\bar{\lambda}_i} \right| \right\}, \quad (47)$$

where if $M_1 = 0$, we take $f = -\infty$ and $g = +\infty$. This completes the proof of Lemma 4.1 and Theorem 4.1. \square

Another set C which satisfies the conclusion of Theorem 4.1 is the set of all $\delta \in \text{Sper } A$ such that the inequalities (34) are satisfied with α replaced by δ and $x_q(\delta)$ has the same sign as $x_q(\alpha)$ for all q such that $x_q \notin \langle \alpha, \beta \rangle$. The connectedness of such a set C is a corollary of Theorem 1.1, as we will explain at the end of this paper.

5 Affine monomial blowings up

In this section we define one of our main technical tools: affine monomial blowing up. We describe a large class of situations in which the valuation ν_α is preserved under blowing up.

Notation. For a subset $J \subset \{1, \dots, n\}$, x_J will stand for the set $\{x_q \mid q \in J\}$.

Let G be an ordered group. For an n -tuple $a = (a_1, \dots, a_n) \in G^n$ of elements of G , we define

$$\text{rat.rk } a = \dim_{\mathbb{Q}} \sum_{j=1}^n \mathbb{Q}a_j \subset G \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Consider a set $J \subset \{1, \dots, n\}$. Fix an element $j \in J$. Let

$$x'_q = x_q \quad \text{if } q = j \text{ or } q \notin J \quad (48)$$

$$= \frac{x_q}{x_j} \quad \text{if } q \in J \setminus \{j\}. \quad (49)$$

Let $x = (x_1, \dots, x_n)$, $x' = (x'_1, \dots, x'_n)$ and $A' = R[x']$. We have a natural ring homomorphism $\pi : A \rightarrow A'$ and the corresponding map of real spectra $\pi^* : \text{Sper } A' \rightarrow \text{Sper } A$.

Remark 5.1 *Since the variables x are monomials in the x' and vice versa, we have*

$$\text{rat.rk}(\nu(x_1), \dots, \nu(x_n)) = \text{rat.rk}(\nu(x'_1), \dots, \nu(x'_n)). \quad (50)$$

Definition 5.1 *The map π is called an **affine monomial blowing up** of $\text{Sper } A$ (along the ideal (x_J)). The choice of the set J , or the ideal (x_J) , is referred to as **the choice of the center of blowing-up** and the choice of $j \in J$ — as **the choice of a coordinate chart**. Finally, let \mathfrak{p} be a prime ideal of A , not containing any of x_1, \dots, x_n . Let ν be a valuation of $\frac{A}{\mathfrak{p}}$. We say that π is an **affine monomial blowing up with respect to ν** if*

$$\nu(x_j) = \min\{\nu(x_q) \mid q \in J\}. \quad (51)$$

*Consider an index $q \in \{1, \dots, n\}$. We say that π is **independent of x_q** if $q \notin J$.*

Remark 5.2 (1) Let \mathfrak{p}' denote the strict transform of \mathfrak{p} in $\text{Spec } A'$. Condition (51) is equivalent to saying that

$$\forall g' \in \frac{A'}{\mathfrak{p}'} \text{ we have } \nu_\delta(g') \geq 0. \quad (52)$$

(2) In the notation of Definition 5.1, suppose that π is independent of x_q . Then $x_q = x'_q$. In this way, we can define the notion of a sequence of affine blowings up, independent of x_q , recursively in the length of the sequence. Namely, if $\pi : A \rightarrow A'$ is a sequence of affine blowings up of length t , independent of x_q , x_q equals the q -th coordinate of A' and $\pi' : A' \rightarrow A''$ is an affine blowing up independent of x_q then we say that $\tilde{\pi} = \pi' \circ \pi$ is a sequence of affine blowings up, independent of x_q .

We consider the coordinates x' as part of the data of the affine monomial blowing up π . An affine monomial blowing up is completely determined by the choice of J and j as above.

We will use the following notation. For a set $E \subset \Gamma_+^n$, let

$$S_E = \{\delta \in \text{Sper}(A) \mid x_i >_\delta 0, i \in \{1, \dots, n\}, \exists a = (a_1, \dots, a_n) \in E \text{ such that} \\ (\nu_\delta(x_1), \dots, \nu_\delta(x_n)) \underset{\circ}{\sim} (a_1, \dots, a_n)\}.$$

and in particular, for $a \in \Gamma_+^n$ we will write

$$S_a = \{\delta \in \text{Sper}(A) \mid x_i >_\delta 0, i \in \{1, \dots, n\}, (\nu_\delta(x_1), \dots, \nu_\delta(x_n)) \underset{\circ}{\sim} (a_1, \dots, a_n)\}$$

We will need the following comparison result which says that blowing up induces a homeomorphism on sets of the form S_E .

Let E be a subset of Γ_+^n . Take a subset $J \subset \{1, \dots, n\}$. Pick a $j \in J$ and consider the affine monomial blowing up

$$\pi : A \rightarrow A' = R[x'],$$

determined by J and j . Assume that π is an affine monomial blowing up with respect to all ν_δ , $\delta \in S_E$ (in other words, $\nu_\delta(x_j) = \min\{\nu_\delta(x_q)\}_{q \in J}$ for all $\delta \in S_E$; we have $\nu_\delta(x'_i) \geq 0$ for all $\delta \in S_E$ and $i \in \{1, \dots, n\}$).

For $a \in \Gamma_+^n$, let a' be the element of Γ_+^n defined by

$$a'_q = a_q \quad \text{if } q \notin J \text{ or } q = j \quad (53)$$

$$a'_q = a_q - a_j \quad \text{if } q \in J \setminus \{j\}. \quad (54)$$

Let $E' = \{a' \mid a \in E\}$ and let $S_{E'}$ denote the corresponding subset of $\text{Sper } A'$. Let

$$\pi^* : \text{Sper } A' \rightarrow \text{Sper } A$$

be the map of real spectra induced by π . It is well known and easy to see that π^* is a homeomorphism away from the zero set $V(x_j)$ of the ideal (x_j) . Since S_E is disjoint from $V(x_j)$, π^* induces a homeomorphism $\pi^*|_{(\pi^*)^{-1}(S_E)} : (\pi^*)^{-1}(S_E) \xrightarrow{\sim} S_E$. For $\delta \in \text{Sper } A \setminus V(x_j)$, let δ' denote the unique preimage of δ in $\text{Sper } A'$.

Proposition 5.1 Take $a \in \Gamma_+^n$ and $\delta \in \text{Sper } A \setminus V(x_j)$. Assume that π is an affine blowing up with respect to ν_δ . Then

$$\nu_\delta = \nu_{\delta'} \quad (55)$$

and

$$\delta' \in S_{a'} \iff \delta \in S_a. \quad (56)$$

Proof: To prove (55), it suffices to show that $R_\delta = R_{\delta'}$. By definition,

$$R_\delta = \left\{ f \in \kappa(\mathfrak{p}_\delta) \mid \exists g \in \frac{A}{\mathfrak{p}_\delta}, |f| \leq g \right\} \quad (57)$$

$$R_{\delta'} = \left\{ f \in \kappa(\mathfrak{p}_\delta) \mid \exists g' \in \frac{A'}{\mathfrak{p}_{\delta'}}, |f| \leq g' \right\}. \quad (58)$$

Since $\frac{A}{\mathfrak{p}_\delta} \hookrightarrow \frac{A'}{\mathfrak{p}_{\delta'}}$, we have $R_\delta \subset R_{\delta'}$. To prove the opposite inclusion, note that, since π is an affine blowing up with respect to ν_δ , we have $\frac{A'}{\mathfrak{p}_{\delta'}} \subset R_\delta$ by (52). Hence, $\forall g' \in \frac{A'}{\mathfrak{p}_{\delta'}}$, $\exists g \in \frac{A}{\mathfrak{p}_\delta}$ such that $|g'| \leq g$. This proves that $R_{\delta'} \subset R_\delta$ as desired.

(56) follows from the equations (48)–(49) and (53)–(55). \square

Corollary 5.1 *Assume that π is an affine blowing up with respect to ν_δ for all $\delta \in S_E$. We have $\pi^*(S_{E'}) = S_E$ and the restriction of π^* to $S_{E'}$ is a homeomorphism.*

Definition 5.2 *The point δ' is called the **transform** of δ . Similarly, a' , E' and $S_{E'}$ are called **transforms** of a , E and S_E , respectively.*

For future reference, we will also define the transform of a \mathbb{Q} -linear relation on $\nu_\delta(x_1), \dots, \nu_\delta(x_n)$. Consider a \mathbb{Q} -linear equality of the form

$$\sum_{i=1}^n \theta_i a_i = 0. \quad (59)$$

Definition 5.3 *The transform of (59) under π is the equality $\sum_{i=1}^n \theta'_i a'_i = 0$, where*

$$\theta'_i = \theta_i \quad \text{if } i \neq j \quad (60)$$

$$= \sum_{q \in J} \theta_q \quad \text{if } i = j. \quad (61)$$

The transforms of all the above objects under sequences of blowings up are defined in the obvious way (that is, as iterated transforms) recursively with respect to the length of the blowing up sequence.

Example: This example shows that Proposition 5.1 (resp. Corollary 5.1) is false without the assumption that π is an affine monomial blowing up with respect to ν_δ (resp. with respect to ν_δ for all $\delta \in S_E$). Let $n = 2$ and $a = ((1, 0), (1, 0)) \in \Gamma^2$. Consider the point $\alpha \in \text{Sper } A$ given by the semi-curve $\alpha = (t^{(1,0)}, t^{(0,1)})$. Let $J = \{1, 2\}$, $j = 1$, $x'_1 = x_1$, $x'_2 = \frac{x_2}{x_1}$. Since $(1, 0) > (0, 1)$ (see the definition of lexicographical in §3), the corresponding blowing up $\pi : A \rightarrow A'$ is not an affine blowing up with respect to ν_α . We have $\mathfrak{p}_{\alpha'} = (0)$, $R_{\alpha'} = R(x'_2)[x_1]_{(x_1)}$, $x_1 > 0$, and the total ordering on the residue field $R(x'_2)$ is given by the inequalities $c < x'_2$ for all $c \in R$ (this information describes α' completely). We have $0 < x'_1 < c < x'_2$ for any positive constant $c \in R$. Moreover, $\nu_{\alpha'}(x'_2) = (0, 0)$, $\nu_{\alpha'}(x'_1) = (1, 0)$, so $\nu_{\alpha'} \neq \nu_\alpha$. We have $\alpha' \in S_{a'}$ but $\alpha \notin S_a$, so the analogues of (56) and Corollary 5.1 do not hold without the assumption that π is an affine monomial blowing up with respect to ν_δ .

6 Desingularization of binomials by monomial blowings up

In this section, we recall and adapt to our context a result from the theory of resolution of singularities—a version of local uniformization of toric varieties by blowings up along smooth centers, or Hironaka’s game.

Proposition 6.1 *Consider two n -tuples $\alpha, \gamma \in \mathbb{N}^n$ and the corresponding monomials x^α and x^γ . Then there exist finitely many sequences $\pi_i : A \rightarrow A'_i$, $1 \leq i \leq s$, of affine monomial blowings up, having the following properties:*

- (1) *For each $i \in \{1, \dots, s\}$, one of x^α and x^γ divides the other in A'_i .*
- (2) *For any prime ideal \mathfrak{p} of A not containing x_1, \dots, x_n , and any valuation ν of $\frac{A}{\mathfrak{p}}$, there exists $i \in \{1, \dots, s\}$ such that π_i is a sequence of affine monomial blowings up with respect to ν .*
- (3) *Assume that there is an index q such that $\alpha_q = \gamma_q = 0$. Then all the blowing up sequences π_i above can be chosen to be independent of x_q .*

Remark 6.1 (1) *Proposition 6.1 is a special case of [20]. We give a proof here since it is much simpler than that of [20].*

(2) *Let ν be a valuation as in Proposition 6.1, and $i \in \{1, \dots, s\}$ an index satisfying the conclusion (2) of the Proposition for ν . Then ν is non-negative on A'_i . If $\nu(x^\alpha) < \nu(x^\gamma)$, we know which of x^α, x^γ divides the other in A'_i , namely, $x^\alpha \mid x^\gamma$ (and not the other way around).*

Proof of Proposition 6.1: (1) and (2) We will define a numerical character $\tau(\alpha, \gamma)$, consisting of a pair of non-negative integers, associated to the *unordered* pair (α, γ) . If one of x^α, x^γ divides the other, there is nothing to prove. Assume that neither of x^α, x^γ divides the other. We will describe a subset $J \subset \{1, \dots, n\}$, such that for any choice of $j \in J$ and the corresponding affine monomial blowing up $A \rightarrow A'$, writing $x^\alpha = x'^{\alpha'}$, $x^\gamma = x'^{\gamma'}$ in the new coordinates, we have

$$\tau(\alpha', \gamma') < \tau(\alpha, \gamma) \quad (62)$$

in the lexicographical ordering. Since for any prime ideal \mathfrak{p} of A not containing x_1, \dots, x_n , and any valuation ν of $\frac{A}{\mathfrak{p}}$, there exists $j \in J$ such that the corresponding affine monomial blowing up $A \rightarrow A'$ is a blowing up with respect to ν (see (51)), Proposition 6.1 (1) and (2) will follow immediately by iterating this procedure.

We start by defining the numerical character $\tau(\alpha, \gamma)$. Let

$$\alpha = (\alpha_1, \dots, \alpha_n) \text{ and} \quad (63)$$

$$\gamma = (\gamma_1, \dots, \gamma_n). \quad (64)$$

Let $\delta_q = \min\{\alpha_q, \gamma_q\}$, $1 \leq q \leq n$; let $\delta := (\delta_1, \dots, \delta_n)$. Let $\tilde{\alpha} := \alpha - \delta$, $\tilde{\gamma} := \gamma - \delta$.

Remark 6.2 *With a view to proving (3) of the Proposition, note that if $\alpha_q = \gamma_q = 0$, we have*

$$\delta_q = \tilde{\alpha}_q = \tilde{\gamma}_q = 0. \quad (65)$$

Interchanging α and γ , if necessary, we may assume that $|\tilde{\alpha}| \leq |\tilde{\gamma}|$ (here and below, $|\cdot|$ stands for the sum of the components). Put

$$\tau(\alpha, \gamma) := (|\tilde{\alpha}|, |\tilde{\gamma}|).$$

Since we are assuming that $x^\alpha \nmid x^\gamma$ in A , we have $\tilde{\alpha} \neq (0, \dots, 0)$ (equivalently, $|\tilde{\alpha}| > 0$). We will now describe a subset $J \subset \{1, \dots, n\}$, such that for any choice of $j \in J$ and the corresponding affine monomial blowing up $\pi : A \rightarrow A'$ along (x_J) , the inequality (62) holds.

Write $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)$, $\tilde{\gamma} = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_n)$. Renumbering the variables, we may assume that there exists a , $1 \leq a < n$, such that $\tilde{\alpha}_j = 0$ for $a < j \leq n$ and $\tilde{\gamma}_j = 0$ for $1 \leq j \leq a$. In other words,

$$\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_a, \underbrace{0, \dots, 0}_{n-a \text{ zeroes}}) \quad (66)$$

$$\tilde{\gamma} = (\underbrace{0, \dots, 0}_a, \tilde{\gamma}_{a+1}, \dots, \tilde{\gamma}_n). \quad (67)$$

We may also assume that

$$\tilde{\alpha}_i > 0 \quad \text{for } 1 \leq i \leq a. \quad (68)$$

Let J denote a minimal subset of $\{1, \dots, n\}$ (in the sense of inclusion), having the following properties:

$$\{1, \dots, a\} \subset J \quad \text{and} \quad (69)$$

$$\sum_{q \in J} \tilde{\gamma}_q \geq |\tilde{\alpha}| \quad (70)$$

Pick a $j \in J$. Let $\pi : A \rightarrow A'$ be the affine monomial blowing up along (x_j) , associated to j and J .

Remark 6.3 *If $\alpha_q = \gamma_q = 0$, (68) implies that $q > a$. Then by the minimality of J and (65), π is independent of x_q .*

We will now write out the monomials x^α and x^γ in the new coordinates and observe that the numerical character τ has strictly decreased. Define the non-negative integers $\tilde{\alpha}'_q$ and $\tilde{\gamma}'_q$, $1 \leq q \leq n$, as follows:

$$\tilde{\alpha}'_q = \tilde{\alpha}_q \quad \text{if } q \neq j \quad (71)$$

$$= 0 \quad \text{if } q = j \quad (72)$$

$$\tilde{\gamma}'_q = \tilde{\gamma}_q \quad \text{if } q \neq j \quad (73)$$

$$= \sum_{q \in J} \tilde{\gamma}_q - |\tilde{\alpha}| \quad \text{if } q = j. \quad (74)$$

Put $\tilde{\alpha}' = (\tilde{\alpha}'_1, \dots, \tilde{\alpha}'_n)$, $\tilde{\gamma}' = (\tilde{\gamma}'_1, \dots, \tilde{\gamma}'_n)$. Let δ' denote the n -vector obtained from δ by adding $|\tilde{\alpha}|$ to the j -th component; that is,

$$\delta' = (\delta_1, \dots, \delta_{j-1}, \delta_j + |\tilde{\alpha}|, \delta_{j+1}, \dots, \delta_n).$$

With these definitions, we have

$$x^\alpha = (x')^{\delta' + \tilde{\alpha}'} \quad (75)$$

$$x^\gamma = (x')^{\delta' + \tilde{\gamma}'}. \quad (76)$$

Put $\alpha' = \delta' + \tilde{\alpha}'$, $\gamma' = \delta' + \tilde{\gamma}'$.

Lemma 6.1 *We have $\tau(\alpha', \gamma') < \tau(\alpha, \gamma)$ in the lexicographical ordering.*

Proof: There are two possibilities: either $j \in \{1, \dots, a\}$ or $j \in \{a+1, \dots, n\}$. If $j \in \{1, \dots, a\}$ then (68), (71) and (72) imply that

$$|\tilde{\alpha}'| = |\tilde{\alpha}| - \tilde{\alpha}_j < |\tilde{\alpha}|. \quad (77)$$

Suppose that $j \in \{a+1, \dots, n\}$. Then by (71),

$$|\tilde{\alpha}'| = |\tilde{\alpha}|. \quad (78)$$

We will prove that $|\tilde{\gamma}'| < |\tilde{\gamma}|$. Indeed, by the minimality of J ,

$$\sum_{q \in J \setminus \{j\}} \tilde{\gamma}_q < |\tilde{\alpha}| \quad (79)$$

(otherwise we could replace J by $J \setminus \{j\}$). Now, by (73), (74) and (79),

$$\begin{aligned} |\tilde{\gamma}'| &= \sum_{\substack{q=a+1 \\ q \neq j}}^n \tilde{\gamma}'_q + \tilde{\gamma}'_j = \sum_{\substack{q=a+1 \\ q \neq j}}^n \tilde{\gamma}_q + \sum_{q \in J} \tilde{\gamma}_q - |\tilde{\alpha}| \\ &= \sum_{q=a+1}^n \tilde{\gamma}_q + \left(\sum_{q \in J \setminus \{j\}} \tilde{\gamma}_q - |\tilde{\alpha}| \right) < \sum_{q=a+1}^n \tilde{\gamma}_q = |\tilde{\gamma}|. \end{aligned} \quad (80)$$

To summarize, (77), (78) and (80) say that

$$(|\tilde{\alpha}'|, |\tilde{\gamma}'|) < (|\tilde{\alpha}|, |\tilde{\gamma}|) = \tau(\alpha, \gamma) \quad (81)$$

in the lexicographical ordering. If $|\tilde{\alpha}'| \leq |\tilde{\gamma}'|$ then according to our definitions

$$\tau(\alpha', \gamma') = (|\tilde{\alpha}'|, |\tilde{\gamma}'|),$$

and the Lemma follows from (81). If $|\tilde{\gamma}'| < |\tilde{\alpha}'|$ then $\tau(\alpha', \gamma') = (|\tilde{\gamma}'|, |\tilde{\alpha}'|) < (|\tilde{\alpha}'|, |\tilde{\gamma}'|)$, and, again, the Lemma follows from (81). This completes the proof of Lemma 6.1. \square

For as long as neither of x^α , x^γ divides the other, we can iterate the construction of Lemma 6.1. Since τ cannot decrease indefinitely, this process must stop after finitely many steps. Therefore after a finite number of steps we will arrive at the situation when one of $x'^{\alpha'}$, $x'^{\gamma'}$ divides the other. In other words, one of x^α , x^γ divides the other in A' .

Of course, the above construction is not unique: at each step we made an arbitrary choice of a coordinate chart. The inequality (62) and hence the final conclusion that one of x^α , x^γ divides the other hold for all the possible choices of j . Now let $\{\pi_i : A \rightarrow A'_i\}_{1 \leq i \leq s}$ be the totality of all the blowing up sequences constructed above for all the possible choices of coordinate charts, such that one of x^α , x^γ divides the other in A'_i for all $i \in \{1, \dots, s\}$ (since the number of choices of coordinate charts is finite at each step and each sequence stops after finitely many steps, the overall set is finite). Since for each valuation ν there always exists a choice of coordinate chart satisfying (51) at each step, the set $\{\pi_i : A \rightarrow A'_i\}_{1 \leq i \leq s}$ satisfies the conclusion of the Proposition. (3) of the Proposition holds by construction according to Remarks 6.2 and 6.3. This completes the proof of Proposition 6.1. \square

Remark 6.4 *Keep the above notation. Take an index $i \in \{1, \dots, s\}$ and consider the blowing up sequence*

$$\pi_i : A \rightarrow A'_i, \quad (82)$$

as above. Let $\pi : B = R[y_1, \dots, y_n] \rightarrow B' = R[y'_1, \dots, y'_n]$ be one of the affine blowings up appearing in the blowing up sequence (82). Let J be the subset of $\{1, \dots, n\}$ such that π is the blowing up along the ideal (y_J) . The definition of J in the proof of Proposition 6.1 depends only on the monomials x^α and x^γ , written in the y coordinates, and makes no mention of any

valuation whatsoever. Now, take a prime ideal \mathfrak{p} of A not containing x_1, \dots, x_n , and let ν be a valuation of $\frac{A}{\mathfrak{p}}$. Let \mathfrak{p}_B be the strict transform of \mathfrak{p} in B . Since, by construction, y_1, \dots, y_n are monomials in x_1, \dots, x_n (with integer exponents), we have $y_1, \dots, y_n \notin \mathfrak{p}_B$, so ν is also a valuation of $\frac{B}{\mathfrak{p}_B}$. Once the set J is defined, the choice of the coordinate chart B' in such a way that π is an affine blowing up with respect to ν depends on ν . Proposition 6.1 (2) asserts that such a choice of coordinate chart is always possible at every step of the definition of the sequence π_i . Namely, we can always choose an index $j \in J$ satisfying $\nu(y_j) = \min_{q \in J} \{\nu(y_q)\}$.

Fix a positive integer $u \leq n$ and rational numbers θ_{jl} , $l \in \{1, \dots, u\}$, $j \in \{1, \dots, n\}$, such that the matrix (θ_{jl}) has rank u . Let $r := n - u$. Let S be a subset of $\text{Sper } A$, such that for all the $\delta \in S$ we have $x_j >_\delta 0$, $j \in \{1, \dots, n\}$, and

$$\sum_{j=1}^n \theta_{jl} \nu_\delta(x_j) = 0, \quad l \in \{1, \dots, u\}. \quad (83)$$

Corollary 6.1 *There exist finitely many sequences $\pi_i : A \rightarrow A'_i = R[x_{i1}, \dots, x_{in}]$, $1 \leq i \leq s$, of affine monomial blowings up having the following property. For each $\delta \in S$ there exists $i \in \{1, \dots, s\}$ such that the following conditions hold:*

(1)

$$\nu_\delta(x_{i,r+1}) = \dots = \nu_\delta(x_{in}) = 0. \quad (84)$$

(2) π_i is a sequence of affine monomial blowings up with respect to ν_δ .

Suppose there is an index $q \in \{1, \dots, n\}$ such that $\theta_{ql} = 0$ for all $l \in \{1, \dots, u\}$. Then all the affine blowing up sequences π_i can be chosen to be independent of x_q .

Proof: We proceed by induction on u . If $u = 0$, we have $r = n$ and there is nothing to prove. Assume that $u > 0$, so $r < n$. Consider the non-trivial \mathbb{Q} -linear relation

$$\sum_{j=1}^n \theta_{j1} \nu_\delta(x_j) = 0, \quad (85)$$

satisfied by all the points $\delta \in S$. For each $j \in \{1, \dots, n\}$ let

$$\alpha_j = \max\{\theta_{j1}, 0\} \quad (86)$$

$$\gamma_j = \max\{-\theta_{j1}, 0\}. \quad (87)$$

Put $\alpha = (\alpha_1, \dots, \alpha_n)$; $\gamma = (\gamma_1, \dots, \gamma_n)$. The equation (85) says that $\nu_\delta(x^\alpha) = \nu_\delta(x^\gamma)$ for all $\delta \in S$. Apply Proposition 6.1 to the pair of monomials x^α, x^γ . Take a point $\delta \in S$ and let $A \rightarrow A' = R[x'_1, \dots, x'_n]$ be a sequence of affine monomial blowings up with respect to ν_δ , appearing in the conclusion of Proposition 6.1 for x^α and x^γ . We have either $\frac{x^\alpha}{x^\gamma} \in A'$ or $\frac{x^\gamma}{x^\alpha} \in A'$. The fact that π is a sequence of blowings up with respect to ν_δ means that

$$\nu_\delta(x'_j) \geq 0 \quad \text{for } 1 \leq j \leq n. \quad (88)$$

Since $\frac{x^\alpha}{x^\gamma}$ is a monomial in x'_1, \dots, x'_n and since

$$\nu_\delta\left(\frac{x^\alpha}{x^\gamma}\right) = \nu_\delta\left(\frac{x^\gamma}{x^\alpha}\right) = 0,$$

(88) implies that one of the $\nu_\delta(x'_q)$ is equal to zero. Renumbering the x'_q , if necessary, we may assume that $\nu_\delta(x'_n) = 0$. Moreover, $\nu_\delta(x'_1), \dots, \nu_\delta(x'_{n-1})$ satisfy $u - 1$ linearly independent \mathbb{Q} -linear relations (namely, the transforms of the relations (83)). Now let

$$\pi_i : A \rightarrow A'_i = R[x_{i1}, \dots, x_{in}], \quad 1 \leq i \leq s,$$

be the totality of affine monomial blowing up sequences satisfying the conclusion of Proposition 6.1, applied to the pair of monomials x^α, x^γ above. For $i, 1 \leq i \leq s$, let S_i denote the set of all the points $\delta \in S$ such that π_i is a sequence of affine monomial blowings up with respect to δ . By Proposition 6.1 (2), we have $S = \bigcup_{i=1}^s S_i$. Let S'_i denote the transform of S_i in $\text{Sper } A'_i$.

At this point, we have constructed finitely many affine monomial blowing up sequences $\pi_i, 1 \leq i \leq s$, having the following properties:

- (1) For each $\delta \in S$ there exists $i \in \{1, \dots, s\}$ such that $\delta \in S_i$.
- (2) For each $i \in \{1, \dots, s\}$ and $\delta \in S_i$ we have

$$\nu_\delta(x_{in}) = 0 \quad (89)$$

and $\nu_\delta(x_{i1}), \dots, \nu_\delta(x_{i,n-1})$ satisfy $u - 1$ linearly independent \mathbb{Q} -linear relations

$$\sum_{j=1}^{n-1} \theta_{jl}^{(i)} \nu_\delta(x_{in}) = 0 \quad (90)$$

(namely, the transforms of the relations (83), from which we may delete $\nu_\delta(x_{in})$ in view of (89)).

Next, for each $i \in \{1, \dots, s\}$, apply the induction hypothesis to A'_i . Since $\nu_\delta(x_{in})$ does not appear in the relations (90), all the subsequent blowings up are independent of x_q . In particular, the equality (89) is preserved by all the subsequent affine monomial blowings up with respect to δ . This completes the proof of the Corollary. \square

Remark 6.5 *Let the notation be as in Corollary 6.1. Take an index $i \in \{1, \dots, s\}$ and consider the blowing up sequence*

$$\pi_i : A \rightarrow A'_i, \quad (91)$$

as in the Corollary. Let $B = R[y_1, \dots, y_n]$ be a polynomial ring appearing at some stage of the blowing up sequence (91). This means that π_i can be written as a composition of affine monomial blowing up sequences of the form $A \xrightarrow{\lambda} B \xrightarrow{\lambda'} A'_i$. Let $S(B)$ denote the subset of S consisting of all the points $\delta \in S$ such that λ is a sequence of affine monomial blowings up with respect to ν_δ . The transforms of the linear relations (83) in B are well defined for all $\delta \in S(B)$ and are independent of the choice of $\delta \in S(B)$. When we are at stage B of the construction of Corollary 6.1, the set J of the next blowing up is defined in a way independent of the choice of $\delta \in S(B)$; the definition of J depends only on the transforms of the linear relations (83) (Remark 6.4 and the proof of Corollary 6.1). For each $j \in J$, let $\mu_j : B \rightarrow B_j$ be the blowing up determined by the choice of the coordinate chart corresponding to j . For $\delta \in S(B)$, we have $\delta \in S(B_j)$ if and only if μ_j is an affine blowing up with respect to ν_δ . Since for each $\delta \in S(B)$ we can always choose an index $j \in J$ satisfying $\nu(y_j) = \min_{q \in J} \{\nu(y_q)\}$, we have $S(B) = \bigcup_{j \in J} S(B_j)$.

Corollary 6.2 *Let the notation be as in Corollary 6.1. Take $\delta \in S$ such that*

$$r = \text{rat.rk}(\nu_\delta(x_1), \dots, \nu_\delta(x_n)) \quad (92)$$

and an index $i \in \{1, \dots, s\}$ such that π_i is a sequence of affine monomial blowings up with respect to δ . Then $\nu_\delta(x_{i1}), \dots, \nu_\delta(x_{ir})$ are \mathbb{Q} -linearly independent.

Proof: This follows immediately from (50) and (92).

7 Connectedness theorems for sets in $\text{Sper } A$.

As usual, let us denote by Γ the ordered group \mathbb{R}_{lex}^n and by Γ_+ the semigroup of non-negative elements of Γ . Let $\omega_{ij}, \theta_{il} \in \mathbb{Q}, i \in \{1, \dots, n\}, j \in \{1, \dots, q\}, l \in \{1, \dots, u\}$ and consider the subset E of Γ_+^n defined by

$$E = \left\{ (a_1, \dots, a_n) \in \Gamma_+^n \mid \sum_{i=1}^n \omega_{ij} a_i > 0, j \in \{1, \dots, q\}, \sum_{i=1}^n \theta_{il} a_i = 0, l \in \{1, \dots, u\} \right\}. \quad (93)$$

Theorem 7.1 *Let $A = R[x_1, \dots, x_n]$ be a polynomial ring. Then the set S_E is connected for the spectral topology of $\text{Sper}(A)$.*

A proof of Theorem 7.1 will be given after a few lemmas.

Lemma 7.1 *Let X be a topological space, normal and compact (not necessarily Hausdorff) and let \mathcal{F} be a filter of non-empty closed connected sets. Then the intersection $C = \bigcap_{F \in \mathcal{F}} F$ is non-empty, closed and connected.*

Proof: That C is non-empty and closed is well known and easy to see. To prove connectedness, suppose $C = X_1 \amalg X_2$, X_1, X_2 closed and non-empty. By normality, there are two open sets $U_1 \supset X_1$, $U_2 \supset X_2$, $U_1 \cap U_2 = \emptyset$. For any $F \in \mathcal{F}$, Let $G(F) = F \setminus (U_1 \cup U_2)$. We have $G(F) \neq \emptyset$ because F is connected. By the compactness of X , we have $\bigcap_{F \in \mathcal{F}} G(F) \neq \emptyset$. Take an element $x \in \bigcap_{F \in \mathcal{F}} G(F)$. Then

$$x \in \bigcap_{F \in \mathcal{F}} G(F) \subset \bigcap_{F \in \mathcal{F}} F = X_1 \amalg X_2,$$

but $x \notin U_1 \cup U_2 \supset X_1 \amalg X_2$, which is a contradiction. \square

Lemma 7.2 *Let B be a finitely generated R -algebra and S a closed connected subset of $\text{Sper}(B)$. Then S is the intersection of a filter of closed connected constructible sets.*

Proof: Consider the filter \mathcal{F} of all the closed connected constructible sets containing S . The filter \mathcal{F} is not empty because it contains the connected component of $\text{Sper } B$ containing S ; such a connected component is constructible by Proposition 1.3 and Remark 1.2.

We want to show that $S = \bigcap_{F \in \mathcal{F}} F$. Suppose that $S \subsetneq \bigcap_{F \in \mathcal{F}} F$ and take a point $\alpha \in \bigcap_{F \in \mathcal{F}} F \setminus S$. There exists a basic open set $U \ni \alpha$ such that $U \cap S = \emptyset$. Then the connected component C of the complement of U , containing S , is a constructible closed connected set containing S , hence a member of \mathcal{F} . But $\alpha \notin C$, contradicting $\alpha \in \bigcap_{F \in \mathcal{F}} F$. \square

In the next lemma, we will use the following notation. Let

$$G = R^s \setminus \left\{ \prod_{j=1}^s x_j = 0 \right\} \cong \text{Maxr } R[x_1, \dots, x_s]_{x_1 \cdots x_s}.$$

Let $f, g : G \rightarrow R_\infty^{n-s}$ be two continuous semi-algebraic functions. Let $D_{f,g}$ and $p_{f,g}$ be as in (28)–(29). Let $\tilde{D}_{f,g}$ be the constructible subset of $\text{Sper } A$ corresponding to $D_{f,g}$ under the bijection between semi-algebraic subsets of R^n and constructible subsets of $\text{Sper } A$, described in the Introduction. Let $\tilde{p}_{f,g} : \tilde{D}_{f,g} \rightarrow \tilde{G}$ be corresponding map of sets in the real spectrum.

Lemma 7.3 (the Projection lemma)

If C_0 is a closed connected subset of $\text{Sper } R[x_1, \dots, x_s]_{x_1 \dots x_s}$, then the set

$$C_{f,g} := \tilde{p}_{f,g}^{-1}(C_0) \quad (94)$$

is connected in $\text{Sper } R[x_1, \dots, x_n]_{x_1 \dots x_s}$.

Proof of Lemma 7.3: According to the previous Lemma, we can write C_0 as the intersection of a filter of closed connected constructible sets $\tilde{F} \in \mathcal{F}$. Take an $\tilde{F} \in \mathcal{F}$. Since \tilde{F} is connected in the real spectrum, its preimage F in the maximal spectrum $R^s \setminus \{\prod_{j=1}^s x_j = 0\}$ is semi-algebraically connected ([5], Proposition 7.5.1, p. 130).

By Proposition 4.1 the closed semi-algebraic subset $p_{f,g}^{-1}(F) \subset D_{f,g}$ is semi-algebraically connected.

Since $p_{f,g}^{-1}(F)$ is semi-algebraic and semi-algebraically connected, $\tilde{p}_{f,g}^{-1}(\tilde{F})$ is also connected for the spectral topology (applying Proposition 7.5.1 of [5] once again). Since

$$C_{f,g} = \bigcap_{\tilde{F} \in \mathcal{F}} \tilde{p}_{f,g}^{-1}(\tilde{F}),$$

$C_{f,g}$ is connected by Lemma 7.1. □

Remark 7.1 By Remark 4.1 the Projection Lemma remains true if one or both strict inequalities in the definition of $\tilde{D}_{f,g}$ and $C_{f,g}$ (28) are replaced by non-strict ones, since the above proof applies verbatim also in that case.

Let the notation be as in Lemma 7.3.

Lemma 7.4 Let $C = \{\delta \in \tilde{p}^{-1}(C_0) \mid x_q > 0, \nu_\delta(x_q) = 0, s < q \leq n\}$. Then C is connected.

Proof of Lemma 7.4: For each $\delta \in C$, we have

$$\nu_\delta(x_{s+1}) = \dots = \nu_\delta(x_n) = 0 \quad (95)$$

by definition of C . (95) is equivalent to saying that the images of $\frac{1}{x_{s+1}}, \dots, \frac{1}{x_n}$ in $A(\delta)$ lie in R_δ , that is, that there exists a polynomial $z \in A$ such that $\frac{1}{x_q} \leq_\delta z$, $s < q \leq n$. Take $j \in \{1, \dots, n\}$ such that $x_j \geq_\delta x_q$, $q \in \{1, \dots, n\}$. Without loss of generality, we may take z to be of the form $z = Nx_j^b$, where $N \in R$ and $b \in \mathbb{N}$. For $N \in R$ with $N \geq 2$, $b \in \mathbb{N}$ and $j \in \{1, \dots, n\}$, let $S_{N,b,j}$ denote the set

$$S_{N,b,j} = \left\{ \delta \in C \mid \frac{1}{x_q} \leq_\delta Nx_j^b, \quad q \in \{s+1, \dots, n\} \right\}.$$

Thus for each $\delta \in C$ there exist $N \in R$ with $N \geq 2$, $b \in \mathbb{N}$ and $j \in \{1, \dots, n\}$ such that $\delta \in S_{N,b,j}$. In other words, $C = \bigcup_{N,b,j} S_{N,b,j}$. Furthermore, take a point $\delta_0 \in C_0$ and let

$(x_1(t), \dots, x_s(t))$ denote the curvette representing δ_0 . Then the curvette $(x_1(t), \dots, x_n(t))$ with $x_q(t) = 1$, $s < q \leq n$, lies in $S_{N,b,j}$ for all the choices of N, b, j as above. Thus to prove the connectedness of C , it is sufficient to prove the connectedness of $S_{N,b,j}$ for all N, b, j .

Fix N, b and j as above. We will now use the Projection Lemma to prove that $S_{N,b,j}$ is connected. First, suppose $j \in \{1, \dots, s\}$. Apply the Projection Lemma with

$$f(x_1, \dots, x_s) = \left(\frac{1}{Nx_j^b}, \dots, \frac{1}{Nx_j^b} \right)$$

and $g = (+\infty, \dots, +\infty)$. Let $D_{f,g}^\bullet$ denote the subset of $D_{f,g}$ defined by replacing the second non-strict inequality in (28) by the strict one. We have

$$S_{N,b,j} = (\tilde{p}_{f,g}^\bullet)^{-1}(C_0),$$

where $\tilde{p}_{f,g}^\bullet = \tilde{p}|_{\tilde{D}_{f,g}^\bullet}$. Then $S_{N,b,j}$ is connected by the Projection Lemma and Remark 7.1.

Next, assume that $j \in \{s+1, \dots, n\}$. First, let

$$\tilde{p}_j : \text{Sper } R[x_1, \dots, x_s, x_j]_{x_1 \dots x_s} \rightarrow \text{Sper } R[x_1, \dots, x_s]_{x_1 \dots x_s}$$

be the natural projection. Let

$$C_{N,b,j} = \left\{ \delta \in \tilde{p}_j^{-1}(C_0) \mid x_j^{b+1} \geq \frac{1}{N} \right\}.$$

Applying the Projection Lemma and Remark 7.1 to $\tilde{p}_{f,g}^\bullet$ with $f = N^{-\frac{1}{b+1}}$, $g = +\infty$, we see that the set $C_{N,b,j}$ is connected. Moreover, $C_{N,b,j}$ is closed since it is the intersection of the preimage of the closed set C_0 under the continuous map \tilde{p}_j and the closed set given by the inequality $x_j^{b+1} \geq \frac{1}{N}$.

Next, consider the natural projection

$$\tilde{p} : \text{Sper } R[x_1, \dots, x_n]_{x_1 \dots x_s x_j} \longrightarrow \text{Sper } R[x_1, \dots, x_s, x_j]_{x_1 \dots x_s x_j}$$

We apply the Projection Lemma once again with s replaced by $s+1$,

$$f(x_1, \dots, x_s, x_j) = \left(\frac{1}{Nx_j^b}, \dots, \frac{1}{Nx_j^b} \right)$$

and $g = (+\infty, \dots, +\infty)$. We have

$$S_{N,b,j} = (\tilde{p}_{f,g}^\bullet)^{-1}(C_{N,b,j}).$$

Then $S_{N,b,j}$ is connected by the Projection Lemma and Remark 7.1. This completes the proof of Lemma 7.4. \square

We need one more general lemma about closures of subsets of $\text{Sper } A$ given by finitely many strict \mathbb{Q} -linear inequalities on $\nu(x_1), \dots, \nu(x_n)$.

Lemma 7.5 *Let E be a subset of Γ_+^n given by finitely many strict \mathbb{Q} -linear inequalities as in (93), but with $u = 0$ (that is, only strict inequalities and no equalities appear in the definition of E). Then E is closed in $\text{Sper } A_{x_1 \dots x_n}$.*

Proof: Let the notation be as in (93) and let $\omega_j = (\omega_{1j}, \dots, \omega_{nj})$, $j \in \{1, \dots, q\}$. Let the notation x^{ω_j} stand for $x_1^{\omega_{1j}} \dots x_n^{\omega_{nj}}$. Then we can write E as

$$E = \{ \delta \in \text{Sper } A_{x_1 \dots x_n} \mid x_i >_\delta 0, i \in \{1, \dots, n\}, \nu_\delta(x^{\omega_j}) > 0 \}.$$

For $\delta \in \text{Sper } A$ such that $x_i >_\delta 0, i \in \{1, \dots, n\}$, saying that $\nu_\delta(x^{\omega_j}) > 0$ is equivalent to saying that $x^{-\omega_j}$ is bounded below by every element of A . This is also equivalent to saying that $x^{-\omega_j}$ is bounded below by every monomial of the form Nx_i^b , $N \in R$, $b \in \mathbb{N}$, $i \in \{1, \dots, n\}$. For $N \in R$, $b \in \mathbb{N}$, $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, q\}$, let

$$S_{N,b,i,j} = \left\{ \delta \in \text{Sper } A_{x_1 \dots x_n} \mid x_i >_\delta 0, i \in \{1, \dots, n\}, x^{-\omega_j} \geq Nx_i^b \right\}.$$

Clearly, $S_{N,b,i,j}$ is relatively closed in $\text{Sper } A_{x_1 \dots x_n}$. We have

$$S_E = \bigcap_{N,b,i,j} S_{N,b,i,j}, \quad (96)$$

hence S_E is also closed. \square

The point of the next two lemmas is to reduce Theorem 7.1 to the case when $u = 0$, that is, to the case when no \mathbb{Q} -linear equalities appear in the definition of E , only strict inequalities.

Pick a subset $J \subset \{1, \dots, n\}$. We want to study blowings up of $\text{Sper } A$ along the ideal (x_J) . There are $\#J$ possible choices of coordinate charts, one for each element $j \in J$. For each $j \in J$, let $\pi_{0j} : A \rightarrow A_j$ denote the affine monomial blowing up, defined by J and j . Let $E_{0j} \subset E$ be defined by

$$E_{0j} = \{a \in E \mid \pi_{0j} \text{ is a blowing up with respect to } a\}$$

and let E'_{0j} denote the transform of E_{0j} under π_{0j} . By definitions, the set E'_{0j} is defined in E by imposing the additional inequalities $a_j \leq a_q$, $q \in J$. As a subset of Γ_+^n , the set E'_{0j} is defined by the linear equalities and strict inequalities, which are the transforms in A_j of the equalities and inequalities (93) (the transforms of the non-strict inequalities $a_q \geq a_j$, $q \in J$, which define E_{0j} inside E , have the form $a'_q \geq 0$; these inequalities hold automatically for $(a'_1, \dots, a'_n) \in \Gamma_+^n$ and need not be taken into account).

Lemma 7.6 *Take two indices $j, \tilde{j} \in J$. Assume that both E_{0j} and $E_{0\tilde{j}}$ are not empty, in other words, that there exist $a, \tilde{a} \in E$ such that*

$$a_j = \min_{q \in J} a_q \quad \text{and} \quad (97)$$

$$\tilde{a}_{\tilde{j}} = \min_{q \in J} \tilde{a}_q. \quad (98)$$

Then there exists a chain of indices $\{j_1, \dots, j_s\} \subset \{1, \dots, n\}$ such that $j = j_1$, $\tilde{j} = j_s$ and $E_{0j_q} \cap E_{0j_{q+1}} \neq \emptyset$ for all $q \in \{1, \dots, s-1\}$.

Proof: Let Γ_\circ denote the subset of Γ_+ consisting of all the n -tuples of real numbers of the form $(d, \underbrace{0, \dots, 0}_{n-1 \text{ zeroes}})$, $d \in \mathbb{R}$. Let Γ_\circ^n denote the subset of Γ_+^n consisting of all the elements of the form (c_1, \dots, c_n) , $c_j \in \Gamma_\circ$. Let $E^\circ = E \cap \Gamma_\circ^n$, $E_{0j}^\circ = E_{0j} \cap \Gamma_\circ^n$, etc.

Given an n -tuple $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ of real numbers and an element $d = (d_1, \dots, d_n)$ of Γ_+ (where $d_q \in \mathbb{R}$), let $d(\epsilon)$ denote the element of Γ_\circ defined by $(d \cdot \epsilon, \underbrace{0, \dots, 0}_{n-1 \text{ zeroes}})$, where

$d \cdot \epsilon = \sum_{q=1}^n d_q \epsilon_q$. If $c = (c_1, \dots, c_n)$ is an element of Γ_+^n , with $c_j \in \Gamma_+$, we will write $c(\epsilon)$ for $(c_1(\epsilon), c_2(\epsilon), \dots, c_n(\epsilon)) \in \Gamma_\circ^n$.

Take two indices $l, \tilde{l} \in J$ and elements $c \in E_{0l}$, $\tilde{c} \in E_{0l} \cap E_{0\tilde{l}}$. Since E_{0l} (resp. $E_{0l} \cap E_{0\tilde{l}}$) is defined by linear equations and inequalities (not necessarily strict), we can choose real numbers

$$\epsilon_1 \gg \epsilon_2 \gg \dots \gg \epsilon_n > 0$$

such that $c(\epsilon) \in E_{0l}$ and $\tilde{c}(\epsilon) \in E_{0l} \cap E_{0\tilde{l}}$. Thus

$$E_{0l} \neq \emptyset \iff E_{0l}^\circ \neq \emptyset. \quad (99)$$

and

$$E_{0l} \cap E_{0\tilde{l}} \neq \emptyset \iff E_{0l}^\circ \cap E_{0\tilde{l}}^\circ \neq \emptyset. \quad (100)$$

As a topological space, identify Γ_\circ^n with \mathbb{R}^n , with the usual Euclidean topology. Since E° is defined in \mathbb{R}_+^n by linear equations and inequalities, it is connected in the Euclidean topology. Now, each E_{0l}° is relatively closed in E° . Hence the conclusion of the Lemma holds with E_q replaced by E_q° . In view of (99)–(100), this completes the proof of the Lemma. \square

Next, we reduce the Theorem to the case when $u = 0$, that is, when E is defined by strict inequalities (and no equalities). Without loss of generality, we may assume that the equalities

$$\sum_{i=1}^n \theta_{il} a_i = 0, \quad l \in \{1, \dots, u\} \quad (101)$$

appearing in the definition of E are linearly independent.

Lemma 7.7 *If Theorem 7.1 is true for $u = 0$ (that is, in the case when there are no equalities in the definition of E) then it is true in general.*

Proof: Assume Theorem 7.1 for $u = 0$. Apply Corollary 6.1 to the set S_E . For each $i \in \{1, \dots, s\}$, let

$$E_i = \{a \in E \mid \pi_i \text{ is a sequence of blowings up with respect to } a\} \quad (102)$$

(cf. Remark 7.2). By Corollary 6.1 (2), $E = \bigcup_{i=1}^s E_i$. Let E'_i denote the transform of E_i under π_i (by definition of E_i , E'_i is well defined). By construction, E'_i is contained in the set $\{a_{r+1} = \dots = a_n = 0\}$, and is defined in $\Gamma_+^n \cap \{a_{r+1} = \dots = a_n = 0\}$ by finitely many strict \mathbb{Q} -linear inequalities — the transforms of the inequalities $\sum_{q=1}^n \omega_{qj} a_q > 0$ under π_i . Let $\iota : \Gamma^r \rightarrow \Gamma^n$

denote the natural inclusion of the set $\{a_{r+1} = \dots = a_n = 0\}$ in Γ^n . Let $E_i^\dagger = \iota^{-1}(E'_i)$. Let \tilde{p}_i denote the natural projection $\tilde{p}_i : \text{Sper } R[x_{i1}, \dots, x_{in}]_{x_{i1} \dots x_{in}} \rightarrow \text{Sper } R[x_{i1}, \dots, x_{ir}]_{x_{i1} \dots x_{ir}}$. The set $S_{E'_i}$ can be written as

$$S_{E'_i} = \left\{ \delta \in \tilde{p}_i^{-1}(E_i^\dagger) \mid x_{ij} >_\delta 0, \nu_\delta(x_{ij}) = 0 \text{ for } r < j \leq n \right\} \rightarrow \text{Sper } R[x_{i1}, \dots, x_{ir}]_{x_{i1} \dots x_{ir}}.$$

Now, $S_{E_i^\dagger}$ is connected by the $u = 0$ case of Theorem 7.1. Then $S_{E'_i}$ is connected by Lemma 7.4. Hence S_{E_i} is connected by Corollary 5.1.

Let N_i denote the number of affine monomial blowings up composing π_i and let

$$N = \max_{1 \leq i \leq s} \{N_i\}.$$

To complete the proof of Lemma 7.7, we will now show that S_E is connected by induction on N . Let $J \subset \{1, \dots, s\}$ be such that the first blowing up in each of the sequences π_i is a blowing up along J (such a J exists according to Remark 6.5).

The set E'_{0j} is defined in Γ_+^n by the linear equalities and strict inequalities, which are the transforms in A_j of the equalities and inequalities (93) (the transforms of the non-strict inequalities $a_q \geq a_j$, $q \in J$, which define E_{0j} inside E , have the form $a'_q \geq 0$; these inequalities hold automatically in Γ_+^n and need not be taken into account). Hence, by the induction assumption, $S_{E'_{0j}} \cong S_{E_{0j}}$ is connected.

Now, Lemma 7.6 shows that $S_E = \bigcup_{j \in J} S_{E_{0j}}$ is connected, as desired. This completes the proof of Lemma 7.7. \square

To complete the proof of Theorem 7.1, it remains to prove

Proposition 7.1 *Assume that $u = 0$ in (93), that is, E is defined by finitely many strict \mathbb{Q} -linear inequalities. Then S_E is connected.*

First proof of Proposition 7.1: Write $S_E = \bigcap_{N,b,i,j} S_{N,b,i,j}$, as in (96). Let

$$S_{N,b} = \bigcap_{j=1}^q \left\{ \delta \in \bigcap_{i=1}^n S_{N,b,i,j} \mid x^{-\omega_j} \geq 1 \right\}.$$

For each N, b the set $S_{N,b}$ is defined by finitely many *monomial* inequalities and the sign conditions $x_t > 0$, $1 \leq t \leq n$. Thus $S_{N,b}$ is connected by Lemma 4.1. If $N \leq N_1$ and $b \leq b_1$ then $S_{N,b} \subset S_{N_1,b_1}$. Then the collection $\{S_{N,b} \mid N \in \mathbb{R}, N > 0, b \in \mathbb{N}\}$ is a filter of connected, closed subsets of $\text{Sper } A_{x_1 \dots x_n}$. Then S_E is connected by Lemma 7.1. This completes the first proof of Proposition 7.1 and hence also of Theorem 7.1. \square

Before giving the second proof of Theorem 7.1, we first prove two connectedness results.

Proposition 7.2 (fiber connectedness) *Fix an element $a = (a_1, \dots, a_n) \in \Gamma_+^n$. The set S_a is connected in $\text{Sper}(A)$.*

Proof of Proposition 7.2: Let $r = \text{rat.rk } a$. Take a subset $J \subset \{1, \dots, n\}$ and let

$$x_J = \{x_q \mid q \in J\}.$$

Let $j \in J$ be such that

$$a_j = \min\{a_q\}_{q \in J} \tag{103}$$

and consider the affine monomial blowing up

$$\pi : A \rightarrow A' = R[x'],$$

determined by J and j :

$$\begin{aligned} x'_q &= x_q, a'_q = a_q && \text{if } q \notin J \text{ or } q = j \\ x'_q &= \frac{x_q}{x_j}, a'_q = a_q - a_j && \text{if } q \in J \setminus \{j\}. \end{aligned}$$

Remark 7.2 *By (103), π is an affine monomial blowing up with respect to δ for all $\delta \in S_a$. In other words, the property of being an affine monomial blowing up with respect to δ depends only on a , not on the particular choice of $\delta \in S_a$. Here and below, we will sometimes say that π is an affine monomial blowing up with respect to a .*

Let (G', a'_1, \dots, a'_n) be the ordered group generated by the a'_i (we have $G' = G$) and let $S_{a'}$ be the corresponding subset of $\text{Sper}(A')$. Then

$$S_a \cong S_{a'} \tag{104}$$

by Proposition 5.1.

We now iterate the above procedure. By Corollaries 6.1 and 6.2 (where we take $S = S_a$), after a succession of such transformations we may assume (after passing to the new coordinates) that a_1, \dots, a_r are \mathbb{Q} -linearly independent and

$$a_{r+1} = \dots = a_n = 0. \tag{105}$$

Again, note that by Remark 7.2, (104) and induction on the number of blowings up required, there is a single sequence of blowings up satisfying the conclusions of Corollary 6.1 which is

a sequence of affine monomial blowings up *simultaneously* for all $\delta \in S_a$. As a_1, \dots, a_r are \mathbb{Q} -linearly independent, there exists a unique point

$$\alpha = (t^{a_1}, \dots, t^{a_r}) \in \text{Sper } R[x_1, \dots, x_r]_{x_1 \dots x_r}$$

such that

$$x_i >_\alpha 0 \quad (106)$$

and $\nu_\alpha(x_i) = a_i$ for $i \in \{1, \dots, r\}$: because of the linear independence of the a_i , the support of α is the zero ideal of $R[x_1, \dots, x_r]$ and the ordering \leq_α of $R(x_1, \dots, x_r)$ is completely described by the inequalities (106) and

$$cx^\gamma < x^\epsilon \quad \text{for all positive } c \in R \iff \sum \gamma_j a_j > \sum \epsilon_j a_j. \quad (107)$$

The fact that a_1, \dots, a_r are \mathbb{Q} -linearly independent implies that (107) imposes a total ordering on the set of monomials and hence on $R(x_1, \dots, x_r)$. The point α is closed in the relative topology of $\text{Sper } R[x_1, \dots, x_r]_{x_1 \dots x_r}$ because α has no non-trivial specializations. Now, let \tilde{p} denote the natural projection $\tilde{p} : \text{Sper } A_{x_1, \dots, x_r} \rightarrow \text{Sper } R[x_1, \dots, x_r]_{x_1 \dots x_r}$. Then

$$S_a = \{ \delta \in \tilde{p}^{-1}(\{\alpha\}) \mid x_q > 0, \nu_\delta(x_q) = 0 \text{ for } r < q \leq n \},$$

hence S_a is connected by Lemma 7.4. This completes the proof of Lemma 7.2. \square

Corollary 7.1 *Let S^\dagger be a connected component of S_E and a an element of Γ_+^n . Then either $S_a \subset S^\dagger$ or $S_a \cap S^\dagger = \emptyset$. There exists a subset $E^\dagger \subset E$ such that $S^\dagger = S_{E^\dagger}$.*

Take $c = (c_1, \dots, c_n), d = (d_1, \dots, d_n) \in E$ such that $c_j = d_j$ for $j \in \{1, \dots, n-1\}$. We define the **segment** $[c, d] \subset E$ as

$$[c, d] = \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ e_n \end{pmatrix} \mid c_n \leq e_n \leq d_n \right\}. \quad (108)$$

Let

$$S_{[c, d]} = \{ \alpha \in S_{(c_1, \dots, c_{n-1}, e_n)} \mid (c_1, \dots, c_{n-1}, e_n) \in [c, d] \}. \quad (109)$$

For an element $c = (c_1, \dots, c_n) \in \Gamma^n$, we will write $c_j = (c_{j1}, \dots, c_{jn})$ with $c_{ji} \in \mathbb{R}$. In this way, we may think of c as an $(n \times n)$ matrix whose j -th row is c_j .

Proposition 7.3 *Take $c, d \in E$ such that :*

- (1) $c_j = d_j$ for $j \in \{1, \dots, n-1\}$
- (2) $c_{n1} < d_{n1}$
- (3) $c_{11}, \dots, c_{n1}, d_{n1}$ are \mathbb{Q} -linearly independent.

Then $S_{[c, d]}$ is connected.

Proof of Proposition 7.3: We start with two Lemmas.

Consider an element $b \in \Gamma_+^n$ such that b_1, \dots, b_n are \mathbb{Q} -linearly independent. Then S_b consists of a single point $\delta(b)$.

Lemma 7.8 *Let U be a basic open set for the spectral topology of $\text{Sper } A$, containing $\delta(b)$. There exists a subset $V \subset \Gamma_+^n$, defined by strict \mathbb{Q} -linear inequalities, such that $b \in V$ and for any $a \in V$, we have $S_a \subset U$.*

Proof: Let $U = \{\alpha \mid f_1(\alpha) > 0, \dots, f_s(\alpha) > 0\}$. Write

$$f_j = \sum_{\gamma \in \mathbb{N}^n} c_{j\gamma} x^\gamma \quad (110)$$

Let $M(j) = c_{j,\gamma(j)} x^{\gamma(j)}$ be the monomial of f_j of smallest valuation (which exists because b_1, \dots, b_n are \mathbb{Q} -linearly independent).

For any j , $1 \leq j \leq s$,

$$\nu_{\delta(b)}(x^{\gamma(j)}) < \nu_{\delta(b)}(x^\gamma) \quad (111)$$

for all γ such that $c_{j\gamma} \neq 0$. Writing $\gamma = (\gamma_1, \dots, \gamma_n)$, $\gamma(j) = (\gamma_1(j), \dots, \gamma_n(j))$, (111) is equivalent to saying that $\sum_{q=1}^n (\gamma_q - \gamma_q(j)) b_q > 0$. Let V be the subset of Γ_+^n , defined by the inequalities

$\sum_{q=1}^n (\gamma_q - \gamma_q(j)) a_q > 0$ for all γ with $c_{j\gamma} \neq 0$. Take an element $a \in V$ and a point $\delta \in S_a$. By construction, each f_j has the same dominant monomial at δ as at $\delta(b)$. Then $S_a \subset U$, as desired.

□

Let $s, r \in \{1, \dots, n-1\}$, $s \leq r$. Consider an element $b \in \Gamma_+^n$ such that

$$b_{r+1} = \dots = b_n = 0$$

and b_1, \dots, b_r are \mathbb{Q} -linearly independent. For $1 \leq j \leq r$, write $b_j = (b_{j1}, \dots, b_{jn})$ with $b_{jq} \in \mathbb{R}$. Consider the $(r \times n)$ -matrix $(b_{jq})_{1 \leq j \leq r, 1 \leq q \leq n}$. Assume that the rows of the $(r \times s)$ -matrix $(b_{jq})_{1 \leq j \leq r, 1 \leq q \leq s}$ are \mathbb{Q} -linearly independent. Define $b^\bullet = (b_1^\bullet, \dots, b_n^\bullet)$ by $b_i^\bullet = b_i, i \neq r+1$ and $b_{r+1}^\bullet = (\underbrace{0, \dots, 0}_s, 1, 0, \dots, 0)$.

Lemma 7.9 *We have $\overline{S_b} \cap S_{b^\bullet} \neq \emptyset$.*

Proof: Pick a point in S_{b^\bullet} , for example,

$$\alpha^\bullet = (t^{b_1}, \dots, t^{b_r}, t^{b_{r+1}^\bullet}, 1, \dots, 1)$$

(here we are representing α^\bullet by a parametrized semi-curve in R^n - see §3). Let

$$U = \{\alpha \mid f_1(\alpha) > 0, \dots, f_q(\alpha) > 0\}$$

be a basic open set such that $U \ni \alpha^\bullet$. It remains to prove that $U \cap S_b \neq \emptyset$. Let $\bar{x} = (x_1, \dots, x_r)$ and $\bar{x} = (x_{r+2}, \dots, x_n)$. Write each f_j as a polynomial in the variables \bar{x} and x_{r+1} , whose coefficients are polynomials in \bar{x} :

$$f_j = \sum_{\gamma = (\bar{\gamma}, \gamma_{r+1})} c_{j\gamma}(\bar{x}) \bar{x}^{\bar{\gamma}} x_{r+1}^{\gamma_{r+1}} \quad (112)$$

(here $c_{j\gamma}(\bar{x})$ is the coefficient in f_j of the monomial $\bar{x}^{\bar{\gamma}} x_{r+1}^{\gamma_{r+1}}$). Let $M(j)$ be the term of smallest valuation among those appearing on the right hand side of (112); such a term is unique because $b_1^\bullet, \dots, b_{r+1}^\bullet$ are \mathbb{Q} -linearly independent. Let $\gamma_{r+1}(j)$ (resp. $\bar{\gamma}(j)$) denote the exponent of x_{r+1} (resp. of \bar{x}) in $M(j)$ and let $c_{j,\gamma(j)} \in R[\bar{x}]$ denote the coefficient of $\bar{x}^{\bar{\gamma}(j)} x_{r+1}^{\gamma_{r+1}(j)}$ in (112), so that $M(j) = c_{j,\gamma(j)} \bar{x}^{\bar{\gamma}(j)} x_{r+1}^{\gamma_{r+1}(j)}$.

To say that $M(j)$ is the monomial of f_j of smallest valuation means that for each j , $1 \leq j \leq q$,

$$\nu(\bar{x}^{\bar{\gamma}(j)}) \leq \nu(\bar{x}^{\bar{\gamma}}) \quad (113)$$

for all $\gamma = (\bar{\gamma}, \gamma_{r+1})$ such that $c_{j\gamma} \neq 0$ and if equality holds in (113) then

$$\gamma_{r+1}(j) < \gamma_{r+1}. \quad (114)$$

Take $\epsilon \in R$ sufficiently small so that for all $j \in \{1, \dots, q\}$ we have

$$\left| c_{j, \gamma(j)}(1, \dots, 1) \epsilon^{\gamma_{r+1}(j)} \right| > \sum_{\substack{\gamma = (\bar{\gamma}, \gamma_{r+1}) \\ \gamma_{r+1} > \gamma_{r+1}(j)}} \left| c_{j, (\bar{\gamma}(j), \gamma_{r+1})}(1, \dots, 1) \epsilon^{\gamma_{r+1}} \right|.$$

Now let $\alpha = (t^{b_1}, \dots, t^{b_r}, \epsilon, 1, \dots, 1)$. We have $\alpha \in U \cap S_b$, as desired. \square

We are now in the position to finish the proof of Proposition 7.3. Suppose $S_{[c,d]}$ were not connected. By Corollary 7.1 we can write

$$[c, d] = B_1 \coprod B_2 \quad (115)$$

and

$$S_{[c,d]} = X_1 \coprod X_2, \quad (116)$$

such that $X_1 = S_{B_1}$ and $X_2 = S_{B_2}$ are open and closed in the relative topology of $S_{[c,d]}$. In view of assumption (1) of Proposition 7.3, (115) induces a decomposition of the segment $[c_n, d_n] = B_{1n} \coprod B_{2n}$.

Consider the segment $[c_{n1}, d_{n1}]$ of the real line.

Lemma 7.10 *Every point $e_{n1} \in [c_{n1}, d_{n1}]$ can be covered by a set $V(1)$, open in the topology induced on $[c_{n1}, d_{n1}]$ by the Euclidean topology of \mathbb{R} , having the following property. Let V be the subset of Γ_+^n defined by*

$$V = \{e \in [c, d] \mid e_{n1} \in V(1)\}. \quad (117)$$

Then all of $S_e, e \in V$ are contained in the same set $X_s, s = 1, 2$.

Lemma 7.10 implies Proposition 7.3. Indeed, assume Lemma 7.10 and consider the decomposition (116). By Lemma 7.10, (116) induces a decomposition $[c_{n1}, d_{n1}] = W_1 \coprod W_2$ into two disjoint open sets in the Euclidean topology of \mathbb{R} , which gives the desired contradiction since $[c_{n1}, d_{n1}]$ is connected.

It remains to prove Lemma 7.10.

Proof of Lemma 7.10:

Case 1 : $e_{n1} \notin \sum_{j=1}^{n-1} \mathbb{Q}c_{j1}$. Let $e_n = (e_{n1}, 0, \dots, 0)$ and $e = (c_1, \dots, c_{n-1}, e_n)$. Without loss of generality, we may assume that $S_e \subset X_1$ (the set S_e consists of a single point). Take a basic open set U of $\text{Sper } A$, containing S_e and disjoint from X_2 . The existence of $V(1)$ with the desired properties follows immediately from Lemma 7.8.

Case 2 : $e_{n1} \in \sum_{j=1}^{n-1} \mathbb{Q}c_{j1}$.

Let (e_{n2}, \dots, e_{nn}) be the unique $(n-1)$ -tuple of real numbers such that letting

$$e_n = (e_{n1}, e_{n2}, \dots, e_{nn}),$$

we have $e_n \in \sum_{j=1}^{n-1} \mathbb{Q}c_{j1}$. Let $\pi : A \rightarrow A'$ be a sequence of affine monomial blowings up with respect to e , constructed in Proposition 6.1 and Corollary 6.1, such that, denoting by e' the transform of e by π , the real numbers $e'_{11}, \dots, e'_{n-1,1}$ are \mathbb{Q} -linearly independent and

$$e'_n = 0.$$

Claim. Assuming $e_n \neq d_n$, we can choose π to be a sequence of blowings up with respect to a small half-open interval of the form $I = [e, h) \subset [c, d]$ with $h_{n1} > e_{n1}$; in other words, for any $b \in [e, h)$, and any $\alpha \in S_b$, we have that ν_α is non-negative on A' .

Proof of Claim: We will go through the procedure of Proposition 6.1 step by step and show by induction on the number of blowings up composing the sequence π that there exists a choice of π as required in the Claim. Recall (Remark 6.4) that the algorithm of Proposition 6.1 consists of choosing the center of blowing up at each step. The algorithm works for every choice of coordinate chart at each step, but the choice of the coordinate chart determines whether the given blowing up is a blowing up with respect to a certain valuation ν .

Let J be the subset of $\{1, \dots, n\}$ prescribed by the algorithm of Proposition 6.1 so that the first blowing up π_1 appearing in π is a blowing up along (x_J) . In the discussion that follows, recall that $c_i = e_i$ for $i \in \{1, \dots, n-1\}$. Since $\dim_{\mathbb{Q}} \sum_{i=1}^n \mathbb{Q}e_{i1} = \dim_{\mathbb{Q}} \sum_{i=1}^n \mathbb{Q}e_i = n-1$, either there exists a unique $j \in J$ such that $e_{j1} < e_{q1}$, $q \in J \setminus \{j\}$, or there exist two indices $j, l \in J$ such that $e_l = e_j < e_q$, $q \in J \setminus \{l, j\}$. In the first case, we have

$$b_j < b_q, \quad q \in J \setminus \{j\} \text{ for every } b = \{b_1, \dots, b_n\} \in [e, h) \quad (118)$$

assuming that $h_{n1} - e_{n1}$ is a sufficiently small positive real number. Use j to determine the coordinate chart for π_1 ; then by (118) π_1 is also a blowing up with respect to every $b \in [e, h)$, as desired.

Next, assume that $e_l = e_j$ are the minimal elements of e_J . Take $h \in [e, d]$ with $h_{n1} - e_{n1} > 0$ sufficiently small so that

$$b_l, b_j < b_q, \quad q \in J \setminus \{l, j\} \text{ for every } b = \{b_1, \dots, b_n\} \in [e, h). \quad (119)$$

We have either $b_j \leq b_l$ for all $b \in [e, h)$ or $b_j \geq b_l$ for all $b \in [e, h)$ (or both). If $b_j \leq b_l$, let j determine the choice of coordinate chart for π_1 ; if $b_j \geq b_l$ for $b \in [e, h)$, let the coordinate chart for π_1 be determined by l . In either case, π_1 is a blowing up with respect to b for all $b \in [e, h)$. Repeating the above procedure at every step of the algorithm of Proposition 6.1, we can choose π as required in the Claim. The Claim is proved.

For any $b \in [e, h)$, we have $S_{b'} \cong S_b$ by Proposition 5.1. Let I' denote the transform of I under π , defined in the obvious way.

Since the natural map $\pi^* : \text{Sper } A' \rightarrow \text{Sper } A$ is continuous and preserves rational rank, we will work with A', c', d', e', I' instead of A, c, d, e, I .

The decomposition (115) induces a decomposition of I and hence also a decomposition

$$I' = B'_1 \amalg B'_2,$$

such that $S_{B'_1}$ and $S_{B'_2}$ are open and closed in the induced topology of $S_{I'}$.

Let $e_n^\bullet = (0, 1, 0, \dots, 0)$, $e^\bullet = (e'_1, \dots, e'_{n-1}, e_n^\bullet)$. Without loss of generality, assume that $e^\bullet \in B'_1$. Let U be a basic open set of $\text{Sper } A$ having non-empty intersection with S_{e^\bullet} and disjoint from $S_{B'_2}$. Now Lemma 7.9 (with $r = n-1$ and $s = 1$) shows that $S_{e^\bullet} \cap \overline{S_{e'}} \neq \emptyset$ and Lemma 7.8 shows that for a sufficiently small interval $V^\dagger(1) = (0, \epsilon)$ of the real line, $\epsilon > 0$, if $b'_{n1} \in V^\dagger(1)$ then $S_{b'} \subset U$. Let $V_+(1) = [0, \epsilon)$ and let $V_+ = \{e' \in [c', d'] \mid e'_{n1} \in V_+(1)\}$. This proves that $\bigcup_{b' \in V_+} S_{b'} \subset S_{B'_1}$.

Coming back to the original interval $[c_{n1}, d_{n1}]$ (that is, before performing the blowing up sequence π), we have shown that all the S_b for $b \in [e, h)$ lie in the same set X_s , $s \in \{1, 2\}$, provided we take h_{n1} sufficiently close to e_{n1} and to the right of e_{n1} . Now, assuming $e_n \neq c_n$, let J be a small half-open interval of the form $(v, e]$ with $v < e$. Repeating the above reasoning

with I replaced by J , we obtain that for $V = (v, h)$, $\bigcup_{b' \in V} S_{b'} \subset S_{B'_1}$, so the open interval $V(1) = (v_{n1}, h_{n1})$ satisfies the conclusion of the Lemma.

This completes the proof of Lemma 7.10 and Proposition 7.3. \square

Second proof of Proposition 7.1: The main idea of this proof is the following. Take two points α and β in S_E . Let

$$a = (a_1, \dots, a_n) : = (\nu_\alpha(x_1), \dots, \nu_\alpha(x_n)) \quad \text{and} \quad (120)$$

$$b = (b_1, \dots, b_n) : = (\nu_\beta(x_1), \dots, \nu_\beta(x_n)). \quad (121)$$

We join α and β by a “staircase” in Γ^n where each stair lies entirely in a connected component of S_E . Two examples at the end of the paper show that we have to do this with some care.

Let $r \in \{1, \dots, n-1\}$. Consider an element $b \in \Gamma_+^n$ such that $b_{r+1} = \dots = b_n = 0$ and b_1, \dots, b_r are \mathbb{Q} -linearly independent. For $1 \leq j \leq r$, write $b_j = (b_{j1}, \dots, b_{jn})$ with $b_{jq} \in \mathbb{R}$. Consider the $(r \times n)$ -matrix $(b_{jq})_{1 \leq j \leq r, 1 \leq q \leq n}$.

Lemma 7.11 *There exists $b' = (b'_1, \dots, b'_n) \in \Gamma_+^n$ having the following properties:*

(1) $b \sim_{\circ} b'$ (in particular, $b'_{r+1} = \dots = b'_n = 0$)

(2) writing $b'_j = (b'_{j1}, \dots, b'_{jn})$, we have that the rows of the $(r \times r)$ -matrix $(b'_{jq})_{1 \leq j \leq r, 1 \leq q \leq r}$ are \mathbb{Q} -linearly independent.

Proof: We will gradually replace b by elements of Γ_+^n , isomorphic to b in the sense of $\mathcal{OGM}(n)$ until we reach an element of Γ_+^n satisfying the conclusion of the Lemma.

Let z be the number of non-zero columns in the matrix $(b_{jq})_{1 \leq j \leq r, 1 \leq q \leq n}$:

$$z = \#\{q \in \{1, \dots, n\} \mid \exists j, 1 \leq j \leq r, \text{ such that } b_{jq} \neq 0\}.$$

Let ℓ be the greatest index in $\{1, \dots, r\}$ such that the column vectors

$$\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{r1} \end{pmatrix}, \dots, \begin{pmatrix} b_{1\ell} \\ b_{2\ell} \\ \vdots \\ b_{r\ell} \end{pmatrix}$$

are \mathbb{R} -linearly independent. We proceed by induction on $z - \ell$. First, suppose $z - \ell = 0$. Since $\ell = z \leq r$, the n -rows b_1, \dots, b_n are obtained from the rows of the matrix $(b_{jq})_{1 \leq j \leq r, 1 \leq q \leq r}$ by adding $n - r$ zeroes at the end. Since b_1, \dots, b_r are \mathbb{Q} -linearly independent by assumption, so are the rows of $(b_{jq})_{1 \leq j \leq r, 1 \leq q \leq r}$. Thus in the case $z - \ell = 0$ we may take $b' = b$ and there is nothing more to prove.

Assume $\ell < z$. Permute the columns of $(b_{jq})_{1 \leq j \leq r, 1 \leq q \leq n}$ by moving all the zero columns to the right of the matrix; this does not change the equivalence class of b with respect to the relation \sim_{\circ} . If this permutation increases the number ℓ , the proof is finished by induction on $z - \ell$. Suppose the number ℓ stays constant after the above permutation of columns. This means that, after the above permutation of columns,

$$\begin{pmatrix} b_{1,\ell+1} \\ b_{2,\ell+1} \\ \vdots \\ b_{r,\ell+1} \end{pmatrix} \text{ is an } \mathbb{R}\text{-linear combination of } \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{pmatrix}, \quad 1 \leq j \leq \ell.$$

Replace $b_{j,\ell+1}$ by 0 for $1 \leq j \leq r$. This operation does not change the equivalence class of b , but decreases the number z of non-zero columns by 1. The result follows by induction on $z - \ell$. \square

Lemma 7.12 *For any connected component S_{E^\dagger} of S_E , there exists $b \in E^\dagger$ such that*

$$\text{rat.rk}(b) = n.$$

Proof: Take $b \in E^\dagger$ such that $\text{rat.rk}(b) = \max \{ \text{rat.rk}(b^\dagger) \mid b^\dagger \in E^\dagger \}$. Suppose

$$\text{rat.rk}(b) = r < n.$$

We will now construct $b^\bullet \in \Gamma_+^n$ such that

$$S_{b^\bullet} \cap \overline{S_b} \neq \emptyset \quad (122)$$

and

$$\text{rat.rk}(b^\bullet) = r + 1. \quad (123)$$

This will contradict the maximality of r .

Let $A \rightarrow A'$ be a sequence of affine monomial blowings up with respect to b , constructed in §6, such that, denoting by b' the transform of b by π , b'_1, \dots, b'_r are \mathbb{Q} -linearly independent and

$$b'_{r+1} = \dots = b'_n = 0.$$

Since the natural map $\pi^* : \text{Sper } A \rightarrow \text{Sper } A'$ is continuous and preserves rational rank, we may replace A by A' and b by b' : if the conditions (122) and (123) are satisfied in $\text{Sper } A'$, they will still be satisfied after applying π^* to everything in sight. From now on, we drop the primes and assume that b_1, \dots, b_r are \mathbb{Q} -linearly independent and $b_{r+1} = \dots = b_n = 0$.

Define $b^\bullet = (b_1^\bullet, \dots, b_n^\bullet)$ by $b_i^\bullet = b_i, i \neq r+1$ and $b_{r+1}^\bullet = \underbrace{(0, \dots, 0)}_r, 1, 0, \dots, 0$. By Lemma

7.9, (122) holds. Since $b \in E$, we have

$$b^\bullet \in E \quad (124)$$

because, by the \mathbb{Q} -linear independence of the rows of the matrix $(b_{jq})_{1 \leq j \leq r, 1 \leq q \leq r}$, the presence of b_{r+1}^\bullet does not affect the strict inequalities defining E : any integer multiple of b_{r+1}^\bullet is infinitesimal in absolute value compared to any non trivial \mathbb{Z} -linear combination of b_1, \dots, b_r . Since $S_b \subset E^\dagger$, (122) implies that $S_{b^\bullet} \subset E^\dagger$. Since $\text{rat.rk}(b^\bullet) = r + 1$, this contradicts the maximality of r . This completes the proof of the Lemma. \square

For $a = (a_1, \dots, a_n)$, write $a_j = (a_{j1}, \dots, a_{jn})$. Take $a \in E$ such that a_{11}, \dots, a_{n1} are \mathbb{Q} -linearly independent, in particular, $\text{rat.rk } a = n$ (such an a exists because the n -tuples (a_{11}, \dots, a_{n1}) such that a_{11}, \dots, a_{n1} are \mathbb{Q} -linearly independent are dense in \mathbb{R}^n).

Now suppose that S_E is not connected and let $S^{(1)}, S^{(2)}$ be two open and closed sets such that $S_E = S^{(1)} \amalg S^{(2)}$ with $S^{(1)}$ containing S_a . Let $E^{(1)}, E^{(2)}$ be the subsets of E such that $S^{(1)} = S_{E^{(1)}}$ and $S^{(2)} = S_{E^{(2)}}$; $E^{(1)}$ and $E^{(2)}$ exist by Corollary 7.1.

Lemma 7.13 *There exists $b \in E^{(2)}$ such that $a_{11}, \dots, a_{n1}, b_{11}, \dots, b_{n1}$ are \mathbb{Q} -linearly independent.*

Proof: According to the preceding lemma, we can find $c \in E^{(2)}$ such that $\text{rat.rk}(c) = n$. Then S_c consists of a single point. Let U be a basic open set containing S_c and such that

$$U \cap S^{(1)} = \emptyset. \quad (125)$$

Let $V \subset \Gamma_+^n$ be a set satisfying the conclusion of Lemma 7.8. Since the n -tuples (b_{11}, \dots, b_{n1}) such that $(a_{11}, \dots, a_{n1}, b_{11}, \dots, b_{n1})$ are \mathbb{Q} -linearly independent are dense in \mathbb{R}^n , there exists $b \in E \cap V$ such that $(a_{11}, \dots, a_{n1}, b_{11}, \dots, b_{n1})$ are \mathbb{Q} -linearly independent. By Lemma 7.8, $S_b \subset U$. By (125), $b \in E^{(2)}$, as desired. \square

Let $a \in E^{(1)}$, $b \in E^{(2)}$ be such that $a_{11}, \dots, a_{n1}, b_{11}, \dots, b_{n1}$ are \mathbb{Q} -linearly independent.

Remark 7.3 If $\lambda, \mu \in \mathbb{Q}$, $\lambda \neq \mu$, then $\lambda a_{11} + (1 - \lambda)b_{11}, \dots, \lambda a_{n1} + (1 - \lambda)b_{n1}, \mu a_{11} + (1 - \mu)b_{11}, \dots, \mu a_{n1} + (1 - \mu)b_{n1}$ are \mathbb{Q} -linearly independent.

In the sequel, let N be a large natural number and $\lambda = \frac{i}{N}$, $\mu = \frac{i+1}{N}$.

Lemma 7.14 For $N \in \mathbb{N}$ sufficiently large we have

$$\begin{pmatrix} (\frac{i}{N}a_{11} + (1 - \frac{i}{N})b_{11}, 0, \dots, 0) \\ \vdots \\ (\frac{i}{N}a_{j-1,1} + (1 - \frac{i}{N})b_{j-1,1}, 0, \dots, 0) \\ (\frac{i+1}{N}a_{j1} + (1 - \frac{i+1}{N})b_{j1}, 0, \dots, 0) \\ \vdots \\ (\frac{i+1}{N}a_{n1} + (1 - \frac{i+1}{N})b_{n1}, 0, \dots, 0) \end{pmatrix} \in E.$$

for all $i \in \{0, \dots, N-1\}$ and all $j \in \{1, \dots, n\}$.

Proof: Since a_{11}, \dots, a_{n1} are \mathbb{Q} -linearly independent, saying that $\sum_{i=1}^n \omega_{il} a_{i1} > 0$, $l \in \{1, \dots, q\}$ is equivalent to saying that

$$\sum_{i=1}^n \omega_{il} a_{i1} > 0, \quad l \in \{1, \dots, q\}.$$

Similarly, we have

$$\sum_{i=1}^n \omega_{il} b_{i1} > 0, \quad l \in \{1, \dots, q\}.$$

The set of n -tuples $(c_{11}, \dots, c_{n1}) \in \mathbb{R}^n$ such that $\sum_{i=1}^n \omega_{il} c_{i1} > 0$, $l \in \{1, \dots, q\}$, is open and convex and contains both (a_{11}, \dots, a_{n1}) and (b_{11}, \dots, b_{n1}) . Then for N sufficiently large we have

$$\begin{aligned} \sum_{i=1}^{j-1} \omega_{il} \left(\frac{i}{N} a_{i1} + \left(1 - \frac{i}{N}\right) b_{i1} \right) + \sum_{i=j}^n \omega_{il} \left(\frac{i+1}{N} a_{i1} + \left(1 - \frac{i+1}{N}\right) b_{i1} \right) = \\ \sum_{i=1}^n \omega_{il} \left(\frac{i}{N} a_{i1} + \left(1 - \frac{i}{N}\right) b_{i1} \right) + \sum_{i=j}^n \omega_{il} \left(\frac{1}{N} a_{i1} - \frac{1}{N} b_{i1} \right) > 0, \end{aligned}$$

$l \in \{1, \dots, q\}$. This completes the proof of the Lemma. \square

For each $i \in \{0, \dots, N-1\}$ and $j \in \{1, \dots, n\}$, consider the pair

$$a(i, j-1) := \begin{pmatrix} (\frac{i}{N}a_{11} + (1 - \frac{i}{N})b_{11}, 0, \dots, 0) \\ \vdots \\ (\frac{i}{N}a_{j-1,1} + (1 - \frac{i}{N})b_{j-1,1}, 0, \dots, 0) \\ (\frac{i+1}{N}a_{j1} + (1 - \frac{i+1}{N})b_{j1}, 0, \dots, 0) \\ \vdots \\ (\frac{i+1}{N}a_{n1} + (1 - \frac{i+1}{N})b_{n1}, 0, \dots, 0) \end{pmatrix}, \quad (126)$$

$$a(i, j) := \begin{pmatrix} (\frac{i}{N}a_{11} + (1 - \frac{i}{N})b_{11}, 0, \dots, 0) \\ \vdots \\ (\frac{i}{N}a_{j,1} + (1 - \frac{i}{N})b_{j,1}, 0, \dots, 0) \\ (\frac{i+1}{N}a_{j+1,1} + (1 - \frac{i+1}{N})b_{j+1,1}, 0, \dots, 0) \\ \vdots \\ (\frac{i+1}{N}a_{n1} + (1 - \frac{i+1}{N})b_{n1}, 0, \dots, 0) \end{pmatrix}. \quad (127)$$

of elements of Γ_+^n .

We can now finish proving the connectedness of S_E . Take $i \in \{0, \dots, N-1\}$ and $j \in \{1, \dots, n\}$. Up to renumbering the components c_1, \dots, c_n , the pair of points $c = a(i, j-1)$ and $d = a(i, j)$ satisfy the hypotheses of Proposition 7.3. Then Proposition 7.3 says that $S_{a(i, j-1)}$ and $S_{a(i, j)}$ are contained in the same connected component of S_E , so $a(i, j-1)$ and $a(i, j)$ belong to the same set $E^{(s)}$, $s = 1, 2$. Put

$$b_{trunc} := \begin{pmatrix} (b_{11}, 0, \dots, 0) \\ \vdots \\ (b_{n1}, 0, \dots, 0) \end{pmatrix}.$$

Since $b \in E^{(2)}$ and $S_b = S_{b_{trunc}}$ by linear independence of b_{11}, \dots, b_{n1} , we must have $b_{trunc} \in E^{(2)}$.

Since $b_{trunc} = a(0, n)$ and $a(i, j-1)$ and $a(i, j)$ belong to the same set $E^{(s)}$ for all i, j , we have $a(i, j) \in E^{(2)}$ for all i, j by induction on $(i, n-j)$ in the lexicographical ordering (note that $a(i, 0) = a(i+1, n)$). Then

$$a(N-1, 0) = a_{trunc} := \begin{pmatrix} (a_{11}, 0, \dots, 0) \\ \vdots \\ (a_{n1}, 0, \dots, 0) \end{pmatrix} \in E^{(2)}$$

so, finally, $a \in E^{(2)}$ which contradicts the fact that a was chosen to lie in $E^{(1)}$. This completes the second proof of Proposition 7.1 (the connectedness of S_E) and with it of Theorem 7.1.

Next, we explain how to use Theorem 7.1 to give another proof of Theorem 4.1. Let the notation be as in the proof of Theorem 4.1. Let C be the set of all $\delta \in \text{Sper } A$ such that the inequalities (34) are satisfied with α replaced by δ and $x_q(\delta)$ has the same sign as $x_q(\alpha)$ for all q such that $x_q \notin \langle \alpha, \beta \rangle$. The set C contains both α and β and none of the polynomials g_i change sign on C . Thus to complete the proof of Theorem 4.1, it remains to prove

Corollary 7.2 (of Theorem 7.1). *The set C is connected.*

Proof: For an index $q \leq l$, replacing x_q by $-x_q$ does not change the problem. Thus we may assume that the sign conditions appearing in the definition of C are all of the form $x_q > 0$ for $q \leq l$. We proceed by induction on $n-l$. If $l = n$ then the Corollary is a special case of Theorem 7.1. Assume that $l < n$ and that the result is known for all the smaller values of $n-l$. Let $C_+ = \{\delta \in C \mid x_n(\delta) > 0\}$, $C_- = \{\delta \in C \mid x_n(\delta) < 0\}$ and $C_0 = \{\delta \in C \mid x_n(\delta) = 0\}$. By the induction assumption, each of C_+ , C_- and C_0 is connected. Take any point $\delta \in C_0$ and a basic open set U of $\text{Sper } A$, containing δ . It is easy to show, by an argument similar to that of Lemma 7.9, that $U \cap C_+ \neq \emptyset$ and $U \cap C_- \neq \emptyset$. Hence $C_0 \subset \bar{C}_+$ and $C_0 \subset \bar{C}_-$. This proves that $C = C_+ \cup C_0 \cup C_-$ is connected, as desired. \square

The next two examples show that the intuition that “convex sets in Γ_+^n give rise to connected sets in $\text{Sper } A$ ” is not completely accurate.

Example 1: Let $n = 2$. Let $c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma^2$. Let $S_{[c,d]}$ be the subset of $\text{Sper } A$, defined in (108) and (109). Let $S_{[c,d]} = S_{[c,d]} \setminus S_c$. Then $S_{[c,d]}$ is not connected: we have $S_{[c,d]} = S_c \amalg S_{[c,d]} = S_c \amalg S_d$ and both S_c and S_d are open and closed. In particular, Proposition 7.3 is false without the assumption that $c_{11}, \dots, c_{n1}, d_{n1}$ are linearly independent. Another way of interpreting this example is that intervals consisting only of rank 1 valuations do not give rise to connected sets in $\text{Sper } A$.

Example 2: In this example, higher rank valuations appear, but, again, $c_{11}, \dots, c_{n1}, d_{n1}$ are \mathbb{Q} -linearly dependent. Let $n = 3$ and take $c_1 = d_1 = e_1 = (1, 0, 0)$, $c_2 = d_2 = e_2 = (0, 1, 0)$, $c_3 = (0, 0, 0)$, $d_3 = (1, 0, 0)$, $e_3 = (e_{31}, e_{32}, e_{33})$, $c_3 \leq e_3 \leq d_3$. Then $c_{11}, c_{21}, c_{31}, d_{31}$ are \mathbb{Q} -linearly dependent and the segment $[c_{31}, d_{31}] = [0, 1]$ is a disjoint union of $V^{(1)} = \{e_{31} = 0\}$ and $V^{(2)} = \{0 < e_{31} \leq 1\}$. The decomposition $[c_{31}, d_{31}] = V^{(1)} \amalg V^{(2)}$ induces a decomposition $S_{[c,d]} = S_{V^{(1)}} \amalg S_{V^{(2)}}$ into two disjoint sets. We claim that $S_{V^{(1)}}$ and $S_{V^{(2)}}$ are both open and closed in the induced topology of $S_{[c,d]}$. To see this, it is sufficient to cover $S_{V^{(1)}}$ by basic open sets of $\text{Sper } A$, disjoint from $S_{V^{(2)}}$, and vice versa.

Let \mathbb{N}_+ denote the set of strictly positive integers. For each $q \in \mathbb{N}_+$, let U_q denote the basic open set of $\text{Sper } A$, defined by the equation $x_3 - x_2^q > 0$. For a point $\delta \in S_{V^{(2)}}$ we have $q\nu_\delta(x_2) = (0, q, 0) < (e_{31}, e_{32}, e_{33}) = \nu_\delta(x_3)$ (see the definition of the lexicographical ordering at the end of section §3), so $x_2^q >_\delta x_3$ and $\delta \notin U_q$. On the other hand, if $\delta \in S_{V^{(1)}}$ then $\nu_\delta(x_3)$ has the form $(0, e_{32}, e_{33})$. Taking $q > e_{32}$, we have $q\nu_\delta(x_2) = (0, q, 0) > (0, e_{32}, e_{33}) = \nu_\delta(x_3)$, so $x_2^q <_\delta x_3$ and $\delta \in U_q$. Thus $S_{V^{(1)}} \subset \bigcup_{q \in \mathbb{N}_+} U_q$ and $U_q \cap S_{V^{(2)}} = \emptyset$ for all $q \in \mathbb{N}_+$. This proves that $S_{V^{(1)}}$ is relatively open in the induced topology.

Next, let W_q denote the basic open set of $\text{Sper } A$, defined by the equation $x_1 - x_3^q > 0$, $q \in \mathbb{N}_+$. For a point $\delta \in S_{V^{(1)}}$ we have $q\nu_\delta(x_3) = (0, qe_{32}, qe_{33}) < (1, 0, 0) = \nu_\delta(x_1)$, so $x_3^q >_\delta x_1$ and $\delta \notin W_q$. On the other hand, if $\delta \in S_{V^{(2)}}$ then $\nu_\delta(x_3)$ has the form (e_{31}, e_{32}, e_{33}) with $e_{31} > 0$. Taking $q > \frac{1}{e_{31}}$, we have $q\nu_\delta(x_3) = (qe_{31}, qe_{32}, qe_{33}) > (1, 0, 0) = \nu_\delta(x_1)$, so $x_3^q <_\delta x_1$ and $\delta \in W_q$. Thus $S_{V^{(2)}} \subset \bigcup_{q \in \mathbb{N}_+} W_q$ and $W_q \cap S_{V^{(1)}} = \emptyset$ for all $q \in \mathbb{Q}_+$. This proves that $S_{V^{(2)}}$ is relatively open in the induced topology, as desired.

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